

Total number of printed pages-8

3 (Sem-4/CBCS) MAT HC 3

2024

**MATHEMATICS**

(Honours Core)

Paper : MAT-HC-4036

**(Ring Theory)**

Full Marks : 80

Time : Three hours

***The figures in the margin indicate full marks for the questions.***

1. Answer the following questions as directed :

1×10=10

- (a) Give an example to show that for two non-zero elements  $a$  and  $b$  of a ring  $R$ , the equation  $ax = b$  can have more than one solution.
- (b) How many nilpotent elements have in an integral domain ?

Contd.



(c) Which of the following statements is not true?

(i)  $\langle 5 \rangle$  is a prime ideal of  $\mathbb{Z}$ .

(ii)  $\langle 5 \rangle$  is a maximal ideal of  $\mathbb{Z}$ .

(iii)  $\langle 5 \rangle$  is a maximal ideal of  $\mathbb{Z}_{20}$ .

(iv)  $\frac{\mathbb{Z}}{5\mathbb{Z}}$  is an integral domain.

(d) Define prime ideal of a ring.

(e) Give example of a commutative ring without zero divisor that is not an integral domain.

(f) Consider the polynomial

$$f(x) = 4x^3 + 2x^2 + x + 4 \text{ and}$$

$$g(x) = 3x^4 + 3x^3 + 3x^2 + x + 4 \text{ in } \mathbb{Z}_5.$$

Compute  $f(x) + g(x)$ .

(g) Write  $f(x) = x^3 + x^2 + x + 1 \in \mathbb{Z}_2[x]$  as a product of irreducible polynomial over  $\mathbb{Z}_2$ .

(h) Which of the following is a primitive polynomial?

(i)  $2x^3 + 4x^2 + 6x + 10$

(ii)  $5x^2 - 30x - 20$

(iii)  $2x^4 + 3x^3 + 5x^2 - 7x + 11$

(iv)  $3x^2 - 3x + 3$

(i) State whether the following statement is true or false :

“A polynomial  $f(x)$  in  $\mathbb{Z}[x]$  which is reducible over  $\mathbb{Z}$  is also reducible over  $\mathbb{Q}$ .”

(j) Choose the correct statement :

(i) Every Euclidean domain is a unique factorization domain.

(ii) Every principal ideal domain is a Euclidean domain.

(iii) Every unique factorization domain is a Euclidean domain.

(iv) Every unique factorization domain is a principal ideal domain.



2. Answer the following questions :  $2 \times 5 = 10$

- (a) If  $a$  and  $b$  are *two* idempotents in a commutative ring, then show that  $a+b-ab$  is also an idempotent element.
- (b) Show that every non-zero element of  $Z_n$  is a unit or a zero divisor.
- (c) Show that every ring homomorphism  $f: Z_n \rightarrow Z_n$  is of the form  $f(x) = ax$  where  $a = a^2$ .
- (d) Find the zeros of  $f(x) = x^2 + 3x + 2$  in  $Z_6$ .
- (e) Let  $D$  be an integral domain and  $a, b \in D$ . If  $\langle a \rangle = \langle b \rangle$ , then show that  $a$  and  $b$  are associates.

3. Answer **any four** questions :  $5 \times 4 = 20$

- (a) The operations  $\oplus$  and  $\otimes$  defined on the set  $Z$  of integers by  $a \oplus b = a + b - 1$  and  $a \otimes b = a + b - ab$ . Show that  $(Z, \oplus, \otimes)$  is a ring with unity.
- (b) Find all ring homomorphism from  $Z \oplus Z$  to  $Z$ .

(c) Let  $R$  be a commutative ring with unity. Show that an ideal  $A$  of  $R$  is prime if and only if the quotient ring  $\frac{R}{A}$  is an integral domain.

(d) Define principal ideal domain. Show that if  $F$  is a field, then  $F[x]$  is a principal ideal domain.  $1+4=5$

(e) Show that every Euclidean domain is a principal ideal domain.

(f) Show that the number of reducible polynomials over  $Z_p$  of the form  $x^2 + ax + b$  is  $\frac{p(p+1)}{2}$ .

4. Answer the following questions :  $10 \times 4 = 40$

(a) (i) Let  $R$  be a commutative ring with unity. Show that the set

$$R[x] = \{a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 / a_i \in R,$$

$n$  is a non-negative integer $\}$  is a ring. Also show that if  $R$  is an integral domain, then  $R[x]$  is also an integral domain.  $5+2=7$



- (ii) Let  $f(x) = 5x^4 + 3x^3 + 1$  and  $g(x) = 3x^2 + 2x + 1$  in  $Z_7[x]$ .

Determine the quotient and remainder upon dividing  $f(x)$  by  $g(x)$ . 3

**Or**

- (i) Show that

$$Z[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in Z\} \text{ and}$$

$$H = \left\{ \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} : a, b \in Z \right\}$$

are isomorphic as ring. 4

- (ii) If  $a, b$  be any two ring elements and  $m$  and  $n$  be any two integers, then show that  $(m.a)(n.b) = (mn).(ab)$  6

- (b) (i) Define maximal ideal of a ring. Let  $A$  be an ideal of a commutative ring with unity  $R$ . Prove that  $\frac{R}{A}$  is a field if and only if  $A$  is maximal. 1+6=7

- (ii) Let  $R$  be a commutative ring and  $A$  be any subset of  $R$ . Show that the nil-radical of  $A$ ,

$$N(A) = \{r \in R / r^n \in A \text{ for some } n \in N\}$$

is an ideal of  $R$ . 3

**Or**

- (i) Let  $\phi$  be a ring homomorphism from  $R$  to  $S$ . Then the mapping

from  $\frac{R}{\ker(\phi)}$  to  $\phi(R)$ , given by

$r + \ker(\phi) \rightarrow \phi(r)$  is an

isomorphism, i.e.,  $\frac{R}{\ker(\phi)} \approx \phi(R)$ . 6

- (ii) Let  $\phi$  be a ring homomorphism from a ring  $R$  to a ring  $S$ . Let  $B$  be an ideal of  $S$ . Then  $\phi^{-1}[B] = \{r \in R : \phi(r) \in B\}$  is an ideal of  $R$ . 4

- (c) If  $F$  is a field and  $p(x) \in F[x]$ , then prove that  $\frac{F[x]}{\langle p(x) \rangle}$  is a field if and only if  $p(x)$  is irreducible over  $F$ .



**Or**

Let  $F$  be a field. If  $f(x) \in F[x]$  and degree  $f(x)$  is 2 or 3, then  $f(x)$  is reducible over  $F$  if and only if  $f(x)$  has a zero in  $F$ . Is the result true when degree  $f(x)$  is greater than 3? Justify.

7+3=10

- (d) In a principal ideal domain, show that an element is irreducible if and only if it is prime. Use this result to show that  $Z[\sqrt{-3}] = \{a + b\sqrt{-3} : a, b \in Z\}$  is not a principal ideal domain. 7+3=10

**Or**

- (i) In a principal ideal domain, show that any strictly increasing chain of ideals  $I_1 \subset I_2 \subset \dots$  must be finite in length. 5
- (ii) Let  $\phi$  be a onto ring homomorphism from a ring  $R$  to a ring  $S$ . Then prove that  $\phi$  is an isomorphism if and only if  $\ker(\phi) = \{0\}$ . 3
- (iii) Determine all ring homomorphism from the ring of integers  $Z$  to itself. 2