Total number of printed pages–24

3 (Sem–5/CBCS) MAT HE 1/2/3

2024

MATHEMATICS

(Honours Elective)

Answer the Questions from any one Option.

OPTION – A

Paper : MAT–HE–5016

(Number Theory)

OPTION – B

Paper : MAT–HE–5026

(Mechanics)

OPTION – C

Paper : MAT–HE–5036

(Probability and Statistics)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

Contd.

OPTION – A

Paper : MAT–HE–5016

(Number Theory)

1. Answer the following questions as directed : $1\times10=10$

(a) State Goldbach conjecture.

(b) If p and q are twin primes, then which of the following statements is true ?

(i) $pq=(p+1)^2-1$

(ii) $pq=(p+1)^2+1$

(iii) $pq=(p-1)^2+1$

(iv) None of the above

(c) Give an example to show that $a^2\equiv b^2 \pmod{n}$ need not imply that $a\equiv b \pmod{n}$.

(d) State whether the following statement is True ***or*** False :

"The polynomial function $f:N\to N$ defined by $f(n)=n^2+n+41$ provides only prime numbers."

(e) Define a pseudoprime number.

(f) Find the sum of all positive divisors of 360.

(g) Which of the following is a perfect number ?

(i) 9

(ii) 10

(iii) 18

(iv) 28

(h) If x is not an integer then find the value of $[x]+[-x]$.

(i) State whether the following statement is True ***or*** False :

"If $\tau(n)$ is an odd integer, then $\sqrt{(n)^{\tau(n)}}$ is not an integer."

(j) Find the number of integers less than 900 and prime to 900.

2. Answer the following questions : $2\times5=10$

(a) Find the remainder when 41^{65} is divided by 7.

(b) If $a \equiv b(\bmod n)$, then show that $a-m \equiv b-m(\bmod n)$, where m is any integer.

(c) Show that any prime of the form $3k+1$ is also of the form $6k+1$, where k is an integer.

(d) If n is an odd positive integer, then prove that $\phi(2n) = \phi(n)$.

(e) For $n \geq 3$, evaluate $\sum_{k=1}^{n} \mu(n!)$

3. Answer ***any four*** questions : 5×4=20

(a) Show that $a \equiv b(\bmod n)$ if and only if a and b have same remainder on division by n.

(b) Show that there are infinite number of primes of the form $4n+3$.

(c) Solve using Chinese Remainder Theorem the simultaneous congruences :

$$x \equiv -2(\bmod 12);\ x \equiv 6(\bmod 10);\ x \equiv 1(\bmod 15)$$

(d) If p is a prime and n is a positive integer, then show that the exponent e such that $p^e / n!$ is atmost $\sum_{i=1}^{\infty}\left[\frac{n}{p^i}\right]$.

(e) Show that the system of linear congruences

$$ax + by \equiv r(\bmod n)$$
$$cx + dy \equiv s(\bmod n)$$

has a unique solution modulo n whenever $\gcd(ad - bc, n) = 1$.

(f) If $n \geq 1$ is an integer, then show that $\sigma(n)$ is odd $\Leftrightarrow n$ is a perfect square or twice a perfect square.

4. (a) State and prove Fermat's theorem. Is the converse of this theorem true ? Justify your answer. 1+5+4=10

OR

(b) (i) Show that every integer $n > 1$, is either a perfect square or the product of a square-free integer and a perfect square. 5

(ii) Let

$$n = a_m(1000)^m + a_{m-1}(1000)^{m-1} + \ldots + a_1(1000) + a_0$$

where a_k's are integers such that

$$0 \le a_k \le 999 \text{ and } T = \sum_{k=0}^{m} (-1)^k a_k.$$

Prove that n is divisible by 7 if and only if T is divisible by 7. 5

5. (a) (i) If p is a prime then show that $(p-1)! \equiv -1 (\bmod p)$. Also verify it for $p = 13$. 4+3=7

(ii) Show that any integer of the form $8^n + 1$ is not a prime. 3

OR

(b) State and prove Fundamental Theorem of Arithmetic. Also find a prime number p such that $2p+1$ and $4p+1$ are also primes. 1+6+3=10

6. (a) (i) For each positive integer $n \ge 1$, prove that

$$\phi(n) = \sum_{d/n} \mu(d)\frac{n}{d} = n\prod_{p/n}\left(1 - \frac{1}{p}\right)$$

7

(ii) If n is the product of a pair of twin primes, prove that

$$\phi(n)\sigma(n) = (n+1)(n-3).$$ 3

OR

(b) (i) If f is a multiplicative arithmetic function, then show that

$$g_1(n) = \sum_{d/n} f(d) \text{ and}$$

$$g_2(n) = \sum_{d/n} \mu(d) f(d)$$

are both multiplicative arithmetic functions. 7

(ii) If n is an even positive integer, then prove that $\phi(2n) = 2\phi(n)$. 3

7. (a) (i) Define Möbius pair. If (f, g) is a Möbius pair and either f or g is multiplicative then show that both f and g are multiplicative. 2+3=5

(ii) If p is a prime number and k is any positive integer, then show that

$$\phi(p^k) = p^k\left(1 - \frac{1}{p}\right).$$ 5

OR

(b) (i) If x and y are real number then show that

$$[x]+[y]\le[x+y]\le[x]+[y]+1.$$ 5

(ii) If $n=p_1^{m_1}.p_2^{m_2}.\ldots.p_r^{m_r}$ where p_i's are distinct primes and

$m_i\in N, m_i\ge 1$ then for each $r\ge 1$ prove that

$$\tau(n)=\prod_{i=1}^{r}(m_i+1).$$ 5

OPTION - B

Paper : MAT–HE–5026

(Mechanics)

1. Answer the following questions : 1×10=10

(a) State the parallelogram law of forces.

(b) Can a force and a couple acting in one plane maintain equilibrium ?

(c) What is the position of the centre of gravity of a uniform triangular lamina ?

(d) Write down the relationship between the co-efficient of friction and the angle of friction.

(e) What is the physical significance of moment of a force about a point ?

(f) Write the expressions for radial and transverse components of acceleration of a particle moving in a plane curve.

(g) Define a conservative force field. Give *one* example.

(h) State the principle of conservation of energy.

(i) A particle moves in a straight line from a distance a towards the centre of force, the force being varies inversely as the cube of the distance. Write down the equation of motion.

(j) State Hooke's law of elasticity.

2. Answer the following questions : 2×5=10

(a) Find the angle between two equal forces each equal to P, when their resultant is a third equal force P.

(b) Two men are carrying a straight uniform bar 6 m long and weighing 30 kg. One man supports it at a distance of 1m from one end and the other man at a distance of 2m from the other end. What weight does each man bear ?

(c) Prove that the centre of gravity of a body is unique.

(d) An impulse I changes the velocity of a particle of mass m from v_1 to v_2. Show that the kinetic energy gained is $\frac{1}{2}I.(v_1 + v_2)$.

(e) State Newton's 2nd law of motion. How does the 2nd law of motion give us a method to measure force ?

3. Answer the following questions : ***(any four)*** 5×4=20

(a) Force $\vec{P}, \vec{Q}, \vec{R}$ acting along $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$, where O is the circum-centre of the triangle ABC, are in equilibrium. Show that

$$\frac{P}{a^2\left(b^2+c^2-a^2\right)}=\frac{Q}{b^2\left(c^2+a^2-b^2\right)}=\frac{R}{c^2\left(a^2+b^2-c^2\right)}$$

(b) Prove that, any system of coplanar forces acting on a rigid body can be reduced ultimately to either a single force or a single couple unless it is in equilibrium.

(c) A uniform ladder rests in limiting equilibrium with the lower end on a rough horizontal plane and its upper end against a smooth vertical wall. If θ be the inclination of the ladder to the vertical, then prove that $tan\,\theta = 2\mu$, where μ is the co-efficient of friction.

(d) A particle is constrained to move along the equiangular spiral $r = ae^{b\theta}$ so that the radius vector moves with constant angular velocity ω. Determine the velocity and acceleration components.

(e) A particle of mass m is acted upon by a force $m\mu\left(x+\frac{a^4}{x^3}\right)$ towards the origin. If it starts from rest at a distance a, show that it will arrive at the origin in time $\frac{\pi}{4\sqrt{\mu}}$.

(f) A particle falls under gravity, supposed constant, in a resisting medium whose resistance varies as the square of the velocity. If the particle starts from rest, derive the expression for velocity of the particle at the end of time t.

4. Answer the following questions : ***(any four)*** 10×4=40

(a) (i) If the resultant of two equal forces inclined at an angle 2θ is twice as great as when they are inclined at an angle 2ϕ, then prove that $\cos\theta = 2\cos\phi$. 5

(ii) P and Q are two like parallel forces. If a couple, each of whose forces is F and whose arm is a in the plane of P and Q, is combined with them, then show that the resultant is displaced through a distance $\frac{Fa}{P+Q}$. 5

(b) (i) Prove that, if three coplanar forces acting on a rigid body be in equilibrium, then they must either all three meet at point, or else all must be parallel to one another. 4

(ii) Force P, Q, R act along the sides $\overline{BC}, \overline{CA}, \overline{AB}$ of the triangle ABC and forces P', Q', R' act along $\overline{OA}, \overline{OB}, \overline{OC}$, where O is the circumcentre, in the senses indicated by the order of the letters. If the six forces are in equilibrium, then show that $P\cos A + Q\cos B + R\cos C = 0$ and $\frac{PP'}{a}+\frac{QQ'}{b}+\frac{RR'}{c}=0$. 6

(c) (i) Define force of friction. What is limiting friction ? State the laws of statical friction and limiting friction. 1+1+3=5

(ii) A body of weight W rests on a rough horizontal plane, λ being the corresponding angle of friction. It is desired to move the body on the plane by pulling it with the help of a string. Find the least angle of friction and the least force necessary. 5

(d) (i) Find the centre of gravity of the arc of the astroid

$$x^{2/3} + y^{2/3} = a^{2/3}$$

lying in the first quadrant. 5

(ii) Find the centre of gravity of the solid formed by revolving $r = a(1+\cos\theta)$ about the x-axis.

5

(e) A particle P, of mass m, moves in a straight line OX under a force $m\mu$(distance) directed towards a point A which moves in the straight line OX with constant acceleration a. Show that the motion of P is simple harmonic of period $\frac{2\pi}{\sqrt{\mu}}$, about a moving centre which is always at a distance $\frac{a}{\mu}$ behind A.

10

(f) One end of an elastic string, whose modulus of elasticity is λ and whose unstretched length is a, is fixed to a point on a smooth horizontal table and the other end is tied to a particle of mass m which is lying on the table. The particle is pulled to a distance where the extension of the string is b and then let go; show that the time of a complete oscillation is

$$2\left(\pi + \frac{2a}{b}\right)\sqrt{\frac{am}{\lambda}}.$$

(g) (i) Show that the path of a point P which possesses two constant velocities u and v, the first of which is in a fixed direction and the second of which is perpendicular to the radius OP drawn from a fixed point O, is a conic whose focus is O and whose eccentricity is $\frac{u}{v}$. 5

(ii) A curve is described by a particle having a constant acceleration in a direction inclined at a constant angle to the target; show that the curve is an equiangular spiral. 5

(h) A particle falls from rest under gravity through a distance x in a medium whose resistance varies as the square of the velocity. If v is the velocity actually acquired by it, v_0 is the velocity it would have acquired had there been no resistance and V is the terminal velocity, show that

$$\frac{v^2}{V_0^2}=1-\frac{1}{2}\frac{v_0^2}{V^2}+\frac{1}{2.3}\frac{v_0^4}{V^4}-\frac{1}{2.3.4}\frac{v_0^6}{V^6}+\ldots\ldots\ldots$$

OPTION - C

Paper : MAT-HE-5036

(Probability and Statistics)

1. Answer the following questions as directed : 1×10=10

(a) Find the total number of elementary events associated to the random experiment of throwing three dice.

(b) Define probability density function for a continuous random variable.

(c) If $P(x)=0.1x,\ x=1$

$\quad\quad 0, \quad$ otherwise

find $P\{x=1 \text{ or } x=2\}$.

(d) If X and Y are two random variables and $var(X-Y)\neq var(X)-var(Y)$ then what is the relation between X and Y?

(e) Can the probabilities of three mutually exclusive events A, B, C as given by $P(A)=\frac{2}{3}$, $P(B)=\frac{1}{4}$ and $P(C)=\frac{1}{6}$ be correct ? If not, give reason.

(f) Mention the relationship among the mean, median and mode of the normal distribution.

(g) Under what condition $cov(X,Y) = 0$?

(h) If X and Y are two independent random variables, then find $var(2x+3y)$.

(i) Write the equation of line of regression of x on y.

(j) If a non-negative real-valued function f, which is the probability density function of the continuous random variable X, is given by $f(x) = 2x, 0 \le x \le 1$ and $P(x \ge a) = P(x > a)$, then find a.

2. Answer the following questions : 2×5=10

(a) If A and B are independent events then show that A and $\bar{B}$ are also independent.

(b) Find k, such that the function f defined by

$$f(x) = kx^2 \text{ when } 0 < x < 1$$
$$0 \quad \text{otherwise}$$

is a probability density function. Also determine $P\left(\frac{1}{3} < x < \frac{1}{2}\right)$.

(c) Determine the binomial distribution for which the mean is 4 and variance is 3.

(d) A random variable X has density function given by

$$f(x) = 2e^{-2x}, x \ge 0$$
$$0, \quad \text{otherwise}$$

then find the moment generating function.

(e) If X and Y are independent random variables with characteristic functions $\varphi_X(\omega)$ and $\varphi_Y(\omega)$ respectively then show that $\varphi_{X+Y}(\omega) = \varphi_X(\omega)\varphi_Y(\omega)$.

3. Answer ***any four*** parts from the following : 5×4=20

(a) A bag contains 5 balls. Two balls are drawn and are found to be white. What is the probability that all are white ?

(b) A random variable X has the function

$$f(x) = \frac{c}{x^2+1}, \text{ where } -\infty < x < \infty, \text{ then}$$

(i) find the value of constant c;

(ii) find the probability that X^2 lies between $\frac{1}{3}$ and 1.

(c) The probability density function of a continuous bivariate distribution is given by the joint density function

$$f(x,y) = x + y, 0 < x < 1, 0 < y < 1$$
$$= 0, \quad \text{elsewhere}$$

Find $E(X), E(Y), var(X), var(Y)$ and $E(XY)$

(d) A coin is tossed until a head appears. What is the expectation of the number of tossed required ?

(e) If X is a Poisson distributed random variable with parameter μ, then show that $E(X) = \mu$ and $var(X) = \mu$.

(f) If X and Y are two independent random variables then show that

$$var(X+Y) = var(X) + var(Y)$$

4. Answer ***any four*** parts from the following : 10×4=40

(a) (i) If $A_1, A_2, \ldots. A_n$ are n mutually exclusive and exhaustive events, then for any event A, prove that

$$P(A) = \sum_{i=1}^{n} P(A_i) P(A/A_i) \text{ and}$$

$$P(A_i/A) = \frac{P(A_i)P(A/A_i)}{P(A)}$$ 5

(ii) A restaurant serves two special dishes, A and B to its customers consisting of 60% men and 40% women, 80% of men order dish A and the rest B. 70% of women order B and the rest A. In what ratio of A to B should the restaurant prepare the two dishes ? 5

(b) (i) Two random variables X and Y have the following joint probability distribution function : 6

$$f(x,y) = 2 - x - y, 0 \le x \le 1, 0 \le y \le 1$$
$$0 \quad , \text{otherwise}$$

Find :

(I) Marginal density function

(II) $E(X)$ and $E(Y)$

(III) Conditional density function

(ii) Determine the Binomial distribution for which the mean is 4 and variance is 3 and find its mode. 4

(c) (i) Find the median of a normal distribution. 5

(ii) A random variable X has density functions given by

$$f(x) = 2e^{-2x}, \quad x \geq 0$$
$$\quad 0 \quad , \quad x < 0$$

Find (I) mean with the help of moment generating function

(II) $P[|X - \mu| > 1]$. 5

(d) (i) A function $f(x)$ of x is defined as follows :

$$f(x) = 0 \quad \text{for } x < 2$$
$$= \frac{1}{18}(3 + 2x) \text{ for } 2 \leq x \leq 4$$
$$= 0, \quad \text{for } x > 4$$

Show that it is a density function. Also find the probability that a variate with this density will lie in the interval $2 \leq x \leq 3$. 5

(ii) A random variable X can assume values 1 and -1 with probability $\frac{1}{2}$ each. Find –

(I) moment generating function

(II) characteristic function. 5

(e) (i) Derive Poisson distribution as a limiting case of binomial distribution. 5

(ii) If 3% of electric bulbs manufactured by a company are defective, using Poisson's destribution, find the probability that in a sample of 100 bulbs exactly 5 bulbs are defective. [Given $e^{-3} = 0.04979$]. 5

(f) (i) Prove that variance of a random variable X can be expressed as the sum of the expectation of the conditional variance and the variance of the conditional expectation is

$$var(X) = E[var(X/Y)] + var[E(X/Y)].$$

5

(ii) If X is a discrete random variable having probability mass function

Mass Point	0	1	2	3	4	5	6	7
$P(X = x)$	0	k	$2k$	$3k$	$4k$	k^2	$2k^2$	$7k^2 + k$

determine :

(a) k

(b) $P(X < 6)$ and

(c) $P(X \geq 6)$ 5

Contd.

(g) (i) Prove that independent variables are uncorrelated. With the help of an example show that converse is not true. 5

(ii) The coefficient of regression of Y on X is $b_{xy} = 1{\cdot}2$.

If $U = \dfrac{X-100}{2}$ and $V = \dfrac{Y-200}{3}$

find b_{VU}. 5

(h) (i) What are the chief characteristics of the normal distribution and normal curve ? 4

(ii) Let X be a random variable with probability density function

$$f(x) = \begin{cases} c\left(1-x^2\right), & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) What is the value of c ? 3

(b) Find the cumulative distribution function of X. 3
