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**3 (Sem–3/CBCS) MAT HC 1**

**2024**

# MATHEMATICS

( Honours Core )

Paper : MAT–HC–3016

***( Theory of Real Functions )***

*Full Marks : 80*

Time : Three hours

***The figures in the margin indicate full marks for the questions.***

1. Answer the following questions : $1\times10=10$

*(a)* Does $\lim_{x\to0} x\sin\left(\frac{1}{x}\right)$ exist ?

*(b)* Define a cluster point of a set $S\subseteq\mathbb{R}$.

*(c)* "If $A\subseteq\mathbb{R}$ and $\phi : A\to\mathbb{R}$ has a limit at a point $a\in\mathbb{R}$, then $\phi$ is bounded on some neighbourhood of *a*." Mention the truth or falsity of this statement.

*Contd.*

(d) Give an example of a function which is discontinuous at every point in $\mathbb{R}$.

(e) Is a uniformly continuous function always continuous?

(f) Mention the points of discontinuity of the greatest integer function $f(x) = [x]$.

(g) Is a function continuous at a point always differentiable at that point?

(h) State Darboux's theorem.

(i) Write Taylor's series for a function $f$, defined on an interval $I$, about a point $a \in I$ when $f$ has all orders of derivatives at $a$.

(j) Write the fourth term in the power series expansion of $\cos x$.

2. Answer the following questions : 2×5=10

(a) Show that $\lim_{x \to a} x^3 = a^3$ by using the $\varepsilon - \delta$ definition of limit.



(b) Prove that a constant function is continuous everywhere.

(c) Applying sequential criterion for limit establish that $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

(d) Find the points of discontinuity of the function $f(x) = \dfrac{(x-3)(x^2+1)}{(x+2)(x-4)}$.

(e) Evaluate the limit $\lim_{x \to \infty} \dfrac{\sqrt{x} - x}{\sqrt{x} + x}$, if it exists.

3. Answer ***any four*** parts of the following : 5×4=20

(a) If $f : D \to \mathbb{R}$ and $a$ is a cluster point of $D$, then prove that $f$ can have only one limit at $a$ if the limit exists.

(b) If $f : I \to \mathbb{R}$, where $I = [a, b]$ be a closed bounded interval, is continuous on $I$, then prove that $f$ has an absolute maximum and an absolute minimum on $I$.

(c) State and prove Bolzano's intermediate value theorem. 1+4=5

(d) If $I$ is a closed and bounded interval and $f : I \to \mathbb{R}$ is continuous on $I$, then prove that $f$ is uniformly continuous on $I$.

(e) State Rolle's theorem and prove it. 1+4=5

(f) Determine whether $x = 0$ is a point of relative extremum of the function $f(x) = \sin x - x$.

4. Answer **any four** parts of the following questions : 10×4=40

(a) If $I = [a, b]$, $f : I \to \mathbb{R}$ is continuous on $I$ and if $f(a) < 0 < f(b)$ or $f(a) > 0 > f(b)$, then prove that there exists a number $c \in (a, b)$ such that $f(c) = 0$.

(b) (i) If $I = [a, b]$ be a closed bounded interval and $f : I \to \mathbb{R}$ is continuous on $I$, then show that $f$ is bounded on $I$. 5



(ii) Let $P(x)$ be a polynomial function of degree $n$. Prove that

$$\lim_{x \to a} P_n(x) = P_n(a).$$ 5

(c) (i) If a function $f$ is uniformly continuous on a bounded subset $A$ of $\mathbb{R}$, then prove that $f$ is bounded on $A$. 5

(ii) Show that the function $f(x) = 1/x$ is uniformly continuous on $I = [1, \infty)$.

(d) (i) If $K > 0$ and the function $f : \mathbb{R} \to \mathbb{R}$ satisfies the condition $|f(x) - f(y)| \leq K|x - y|$, for all real numbers $x$ and $y$, then show that $f$ is continuous at every point $c \in \mathbb{R}$. Further, from it conclude that $f(x) = |x|$ is continuous at every point $c \in \mathbb{R}$. 4+2=6

(ii) Show that the function $f$ defined by

$$f(x) = \frac{e^{1/x} - 1}{e^{1/x} + 1}, \text{ if } x \neq 0$$

$$= 0 \quad , \text{ if } x = 0$$

is discontinuous at $x = 0$ 4

(e) State Caratheodory's theorem and prove it completely. Apply this theorem to show that $f(x) = 2x^3 + 1$ is differentiable at $a \in \mathbb{R}$ and that $f'(a) = 6a^2$. 2+4+4=10

(f) If $f : I \to \mathbb{R}$ is differentiable on the interval $I$, then prove that

(i) $f$ is increasing iff $f'(x) \geq 0, \forall x \in I$.

(ii) $f$ is decreasing iff $f'(x) \leq 0, \forall x \in I$.

Hence prove that

$$f(x) = x^3 - \frac{9}{2}x^2 + 6x - 1$$

is decreasing in the interval $(1, 2)$.

3½+3½+3=10



(g) (i) Find the derivative of $f(x) = \sin\sqrt{x}$ using the definition of derivative. 4

(ii) State and prove Cauchy's Mean Value Theorem. 2+4=6

(h) (i) Evaluate : $\lim_{x \to 0} \frac{x^2 - \sin^2 x}{x^4}$. 5

(ii) Prove that $e^{\pi} > \pi^{e}$. 5

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