

Total number of printed pages-7

3 (Sem-3/CBCS) MAT HC 2

2024

MATHEMATICS

(Honours Core)

Paper : MAT-HC-3026

(Group Theory-I)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions as directed :

$1 \times 10 = 10$

(a) "The set S of positive irrational numbers together with 1 is a group under multiplication." Justify whether it is true or false.

(b) "In the set of integers, subtraction is not associative." Justify the statement.

(c) "Product of two subgroups of a group is again a subgroup." State whether true or false.

Contd.

(d) In the group \mathbb{Z}_{12} , find the order of 6.

(e) Write all the generators of \mathbb{Z}_8 .

(f) List all the elements of the group $\frac{\mathbb{Z}}{4\mathbb{Z}}$.

(g) Give the statement of Cayley's theorem.

(h) Write the following permutation as product of 2-cycles :

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 4 & 6 & 5 & 7 & 2 \end{pmatrix}$$

(i) State whether the following statement is true or false :

"If the homomorphic image of a group is Abelian, then the group itself is Abelian."

(j) Give the statement of second isomorphism theorem.

2. Answer the following questions : $2 \times 5 = 10$

(a) Prove that in a group G , for any elements a and b and any integer n ,

$$(a^{-1}ba)^n = a^{-1}b^na.$$

(b) Show that in a group G , right and left cancellation laws hold.

(c) Define centre of a group G and give an example.

(d) What is meant by cycle of a permutation? Give an example.

(e) If ϕ is a homomorphism from a group G onto a group \bar{G} , then show that ϕ carries the identity element of G to the identity element of \bar{G} .

3. Answer **any four** questions : $5 \times 4 = 20$

(a) Let G be a group and H be a non-empty finite subset of G . Prove that H is a subgroup of G if and only if H is closed under the operation in G .

(b) Let G be a group and H be a subgroup of G . For an element a in G , prove that $aH = H$ if and only if a is in H .

(c) Let G be a finite group and H be a subgroup of G . Prove that $|H|$ divides $|G|$.

(d) Prove that in a finite group, the number of elements of order d is divisible by $\phi(d)$.

(e) Define external direct product of a finite collection of groups. List all elements of $U(8) \oplus U(10)$ and find $|U(8) \oplus U(10)|$.
 $2+2+1=5$

(f) Let ϕ be a homomorphism from a group G to a group \bar{G} . If \bar{K} is a normal subgroup of \bar{G} , prove that $\phi^{-1}(\bar{K}) = \{k \in G \mid \phi(k) \in \bar{K}\}$ is a normal subgroup of G .

Answer **either** (a) **or** (b) from each of the following questions (Q.4 to Q.7) :
 $10 \times 4 = 40$

4. (a) Describe the elements of D_4 , the symmetrics of a square. Write down a complete Cayley's table for D_4 . Show that D_4 forms a group under composition of functions. Is D_4 an Abelian group?
 $2+3+4+1=10$

(b) Let G be a group and H be a non-empty subset of G . Prove that H is a subgroup of G if and only if $a.b^{-1}$ is in H whenever a and b are in H . Also, write all the subgroups of the group of integers \mathbb{Z} .
 $8+2=10$

5. (a) (i) Let a be an element of order n in a group G and let k be a positive integer. Then prove that

$$\langle a^k \rangle = \langle a^{gcd(n,k)} \rangle \text{ and}$$

$$|\langle a^k \rangle| = \frac{n}{gcd(n,k)}$$

(ii) Prove that in a finite cyclic group, the order of an element divides the order of the group.
 $8+2=10$

(b) Prove that every subgroup of a cyclic group is cyclic. Also, if $|G| = n$, then show that the order of any subgroup of G is a divisor of n ; and for each positive divisor k of n , the group G has exactly one subgroup of order k , which is $\langle a^{n/k} \rangle$.
 $5+2+3=10$

6. (a) Prove that every group is isomorphic to a group of permutations.

(b) Prove that the order of a permutation of a finite set written in disjoint cycle form is the least common multiple of the lengths of the cycles.

7. (a) (i) Let ϕ be a group homomorphism from a group G to a group \bar{G} .

Prove that $\frac{G}{\text{Ker } \phi}$ is isomorphic to

$\phi(G)$.

(ii) Prove that $\frac{\mathbb{Z}}{\langle n \rangle} \cong \mathbb{Z}_n$. 7+3=10

(b) Let ϕ be an isomorphism from a group G onto a group \bar{G} . Prove that :

(i) for every integer n and for every group element a in G ,

$$\phi(a^n) = [\phi(a)]^n.$$

(ii) $|a| = |\phi(a)|$ for all a in G .

(iii) if G is finite, then G and \bar{G} have exactly the same number of elements of every order.

$$4+3+3=10$$