3 (Sem–6/CBCS) MAT HE 6

2025

MATHEMATICS

(Honours Elective)

Paper : MAT–HE–6066

(Group Theory–II)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions as directed : $1\times10=10$

   (a) Which one of the following is true ?

   (i) $Z_3 \oplus Z_5 \approx Z_{15}$

   (ii) $Z_3 \oplus Z_6 \approx Z_{18}$

   (iii) $Z_2 \oplus Z_4 \approx Z_8$

   (iv) All of the above

   (b) Define internal direct product of two groups.

(c) Let G be a group and H, K be two subgroups of G. Mention the condition under which $HK = \{hk : h \in H, k \in K\}$ is a subgroup of G.

(d) List the elements of $U_8(35)$.

(e) State whether the following statement is true ***or*** false :
"Any two cyclic group of same order are isomorphic."

(f) Give reason why '21=2+3+3+3+3+7' is not representing class equation of a group of order 21.

(g) Which of the following statement is not true ?

(i) $U(105) \approx Z_2 \oplus Z_4 \oplus Z_6$

(ii) $U(105) \approx U(3) \oplus U(5) \oplus U(7)$

(iii) $U(105) \approx U(21) \oplus U(5)$

(iv) None of the above

(h) Define Sylow p-subgroup of a finite group.

(i) In D_4, write the conjugacy class of R_0.

(j) Prove or disprove : "There is a simple group of order 150."

2. Answer the following questions : 2×5=10

(a) List all Abelian groups (up to isomorphism) of order 360.

(b) Construct $\frac{Z}{N}$, where Z is the additive group of integers and $N = \{5n : n \in Z\}$.

(c) Prove or disprove : $U(10) \approx U(12)$.

(d) Show that a group of order 40 has a unique normal subgroup of order 5.

(e) State Sylow Test for non-simplicity of groups.

3. Answer ***any four*** questions : 5×4=20

(a) Define inner automorphism of groups. Show that the inner automorphism of a group G is a subgroup of G. 1+4=5

(b) Show that every quotient group of a cyclic group is cyclic. Is the converse true ? Justify your answer. 2+3=5

(c) State Sylow's Third Theorem. Use it to show that *any two* Sylow 2-subgroups of the symmetric group S_3 are conjugate. 2+3=5

(d) Let G be a finite group and $C(a)$ be the centralizer of the element a of G. Show that $|cl(a)| = \frac{|G|}{|C(a)|}$.

(e) Show that an integer of the from $2n$, where n is an odd integer greater than one is not the order of a simple group.

(f) If m, n are two relatively prime integers, then show that $U(mn) \approx U(m) \oplus U(n)$.

4. Answer the following questions : 10×4=40

(a) Let G be a group and $g, h \in G$.

(i) If $z \in Z(G)$, then show that the inner automorphism induced by g and zg are identical.

(ii) If g and h induce the same inner automorphism of the group G then prove that $h^{-1}g \in Z(G)$.

(iii) If ϕ is an automorphism of the Dihedral group D_4 such that $\phi(R_{90}) = R_{270}$ and $\phi(V) = V$, then determine $\phi(D)$ and $\phi(H)$.

4+3+3=10

Or

Show that the order of an element of $G_1 \oplus G_2 \oplus \ldots \oplus G_n$ is the least common multiple of the orders of the components of the element. Hence find the number of elements of order 5 in $Z_{25} \oplus Z_5$. 5+5=10

(b) Show that if a group G is internal direct product of a finite number of subgroups $H_1, H_2, \ldots H_n$, then G is isomorphic to the external direct product of $H_1, H_2, \ldots H_n$. 10

Or

(i) Show that a subgroup H of a group G is normal subgroup of G if and only if $xHx^{-1} \subseteq H$, $\forall x \in G$. 4

(ii) For any group G, show that $\frac{G}{Z(G)} \approx Inn(G)$. 6

(c) Let G be a finite group and let p be a prime number that divides $|G|$, then show that G has an element of order p. 10

Or

(i) Define conjugacy of an element of a group. Show that conjugacy is an equivalence relation on a group. 1+4=5

(ii) If a is a element of a group G, then show that $cl(a)=\{a\}$ and only if $a \in Z(G)$. Hence show that G is Abelian if and only if $cl(a)=\{a\}$. 3+2=5

(d) (i) Let G be a group and H be a subgroup of G. Let S be the group of all permutations of the left cosets of H in G. Show that there is a homomorphism from G into S whose kernel lies in H and contains every normal subgroup of G that is contained in H. 5

(ii) If a finite non-Abelian simple group G has a subgroup of index n, then G is isomorphic to a subgroup of A_n. 5

Or

Let G be an Abelian group of prime-power order and let a be an element of maximum order in G. Then show that G can be written in the form $\langle a \rangle \times K$. 10
