3 (Sem–4/CBCS) MAT HC 2

2025

MATHEMATICS

(Honours Core)

Paper : MAT–HC–4026

(Numerical Methods)

Full Marks : 60

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions : 1×7=7

   (i) Name the underlying theorem on which the Bisection method is based upon ?

   (ii) Show that $1 + \Delta = E$

   (iii) Show that $\nabla = 1 - E^{-1}$

   (iv) Out of the following two rules, which will give the accurate result of $\int_{-1}^{1} x^2 \, dx$?

   (a) Trapezoidal Rule

(b) Simpson's Rule

Give reason for your answer.

(v) Write the name of a method for solving a first-order non-linear ordinary differential equation with a given initial condition.

(vi) Identify the correct statement from the following :

(a) Lagrange's interpolation formula can be used only if unequally spaced arguments are given.

(b) Lagrange's interpolation formula can be used only if equally spaced arguments are given.

(c) Lagrange's interpolation formula can be used for both equally and unequally spaced arguments.

(d) Lagrange's interpolation formula can be used only if equally spaced function values are given.

(vii) Write *two* advantages of interpolation.

2. Answer the following questions : 2×4=8

(i) Given $f(x) = x^3 - 4x^2 + 5x - 2$. Can bisection method be applied for this function in any sub-interval interval of $(0,2)$? Will it be applicable for any sub-interval of $(1,3)$? Give proper justification.

(ii) What do you mean by fixed point of a function ? Find the fixed point (s) of the function $f(x) = 2x(1-x)$.

(iii) With proper reasoning state whether fixed point iteration method can be applicable for finding solution of the equation :

$$2x = \sin x + 5.$$

(iv) Find the approximate value of $\int_1^2 \frac{1}{x} dx$ using Trapezoidal rule.

3. Answer **any three** questions : 5×3=15

(i) Determine the LU decomposition for the matrix A where

$$A = \begin{bmatrix} 1 & 4 & 3 \\ 2 & 7 & 9 \\ 5 & 8 & -2 \end{bmatrix}.$$

(ii) Determine the Newton's form of the interpolating polynomial for the following data set :

x	0	1	2
y	2	-1	4

Use this polynomial to estimate the values of y when $x = 1{\cdot}5$.

(iii) Consider the function $f(x) = \sin x$. Construct the Lagrange's form of the interpolating polynomial for f passing through the points $(0, \sin 0)$, $\left(\frac{\pi}{4}, \sin\frac{\pi}{4}\right)$ and $\left(\frac{\pi}{2}, \sin\frac{\pi}{2}\right)$. Use this obtained polynomial to estimate $\sin\left(\frac{\pi}{3}\right)$. What is the error in this approximation ?

3+1+1=5

(iv) What do you mean by degree of precision of a quadrature rule ? Determine values for the co-efficients A_0, A_1 and A_2 so that the quadrature formula

$$I(f) = \int_{-1}^{1} f(x)dx = A_0 f\left(-\frac{1}{2}\right) + A_1 f(0) + A_2 f\left(\frac{1}{2}\right)$$

has degree of precision at least 2.

1+4=5

(v) Derive Euler's method for approximating the solution to the initial value problem,

$$y'(t) = f(t, y(t)), \; a \le t \le b$$

$$y(a) = \alpha$$

4. Answer ***any three*** questions : 10×3=30

(i) Verify that the equation $x^4 - 18x^2 + 45 = 0$ has a root on the interval $(1,2)$. Perform three iterations of Newton's Method with initial approximation $p_0 = 1$. Given that the exact value of the root is $x = \sqrt{3}$, compute the absolute error in the approximation obtained. What is the apparent order of convergence ?

1+6+2+1=10

(ii) Develop the method of false position for approximating the zeros of continuous functions. Provide an appropriate stopping condition for the method of false position. Given that the function $f(x) = x^3 + 2x^2 - 3x - 1$ has a simple zero on $(1,2)$, perform three iterations of the method of false position to demonstrate the general procedure.

5+2+3=10

(iii) If $x_0, x_1, \ldots, x_n$ are $(n+1)$ distinct points and f is defined at $x_0, x_1, \ldots, x_n$, then prove that there exists a unique interpolating polynomial P of degree at most n such that P interpolates f.

Next, consider the data set

x	-1	0	1	2
y	5	1	1	11

Show that the polynomials

$f(x) = x^3 + 2x^2 - 3x + 1$ and

$g(x) = \frac{1}{8}x^4 + \frac{3}{4}x^3 + \frac{15}{8}x^2 - \frac{11}{4}x + 1$

both interpolate all the data. Explain why this does not contradict the uniqueness part of the result you have just proved. 5+2+3=10

(iv) Let $x_0, x_1, x_2, \ldots, x_n$ be $(n+1)$ distinct points in $[a,b]$. If f is continuous on $[a,b]$ and f has $(n+1)$ continuous derivatives on (a,b), then prove that for each $x \in [a,b]$, there exists a $\xi(x) \in [a,b]$, such that

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1) \ldots\ldots (x - x_n),$$

where P is the interpolating polynomial.

(v) If f is continuous on $[a,b]$, g is integrable on $[a,b]$ and $g(x)$ does not change sign on $[a,b]$, then prove that there exists a number $\xi \in [a,b]$ such that $\int_a^b f(x)\,g(x)dx = f(\xi)\int_a^b g(x)dx$.

Use the above result to determine the error term associated with the Newton-Cotes formula given as,

$$I(f) = I_1(f) + \int_a^b f[a,b,x](x-a)(x-b)dx$$

where $I(f) = \int_a^b f(x)dx$,

$I_1(f) = \int_a^b P_1(x)dx$ and

P_1 is unique linear polynomial that interpolates the integrand at $x = a$ and $x = b$. 7+3=10

(vi) Derive a formula for approximating the first derivative of an arbitrary function at $x = x_0$ by interpolating at $x = x_0 + h$ and $x = x_0 - h$.

Also, derive a formula for approximating the second derivative of an arbitrary function at $x = x_0$ by interpolating at $x = x_0 + h$, $x = x_0$ and $x = x_0 - h$.

5+5=10