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3 (Sem-3/CBCS) MAT HC 1

2025

**MATHEMATICS**

(Honours)

Paper : MAT-HC-3016

**( Theory of Real Functions )**

Full Marks : 80

Time : Three hours

**The figures in the margin indicate full marks for the questions.**

1. Answer the following as directed :  $1 \times 10 = 10$

(a) Write the number of solutions of the equation  $\ln(x) = x - 2$ .

(b) If  $f(x) = |x^3|$  for  $x \in \mathbb{R}$ , then find  $f'(x)$  for  $x \in \mathbb{R}$ .

(c) Between any two roots of the function  $f(x) = \sin x$ , there is at least \_\_\_\_\_ root of the function  $f(x) = \cos x$ .

(Fill in the blank)

- (d) The derivative of an even function is \_\_\_\_\_ function. (Fill in the blank)
- (e) "Every polynomial of odd degree with real coefficients has at least one real root." Is this statement true **or** false?
- (f) Write the points of continuity of the function  $f(x) = \cos \sqrt{1+x^2}$ ,  $x \in \mathbb{R}$ .
- (g) If a function  $f: (a, \infty) \rightarrow \mathbb{R}$  is such that  $\lim_{x \rightarrow \infty} xf(x) = L$ , where  $L \in \mathbb{R}$ , then  $\lim_{x \rightarrow \infty} f(x) = ?$
- (h) Write the cluster points of  $A = (0, 1)$ .
- (i) Is the function  $f(x) = x \sin\left(\frac{1}{x}\right)$  continuous at  $x=0$ ?
- (j) Find  $\lim_{x \rightarrow 2} \frac{x^3 - 4}{x^2 + 1}$ .

2. Answer the following questions :  $2 \times 5 = 10$

- (a) Suppose  $f: \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = x^2 \sin \frac{1}{x^2}$ , for  $x \neq 0$  and  $f(0) = 0$ . Is  $f'$  bounded on  $[-1, 1]$ ?
- (b) Give an example with justification that a function is 'continuous at every point but whose derivative does not exist everywhere'.
- (c) Let  $f$  be defined for all  $x \in \mathbb{R}$ ,  $x \neq 3$  by  $f(x) = \frac{x^2 + x - 12}{x - 3}$ . Can  $f$  be defined at  $x=3$  in such a way that  $f$  is continuous at this point?
- (d) Show that  $f(x) = x^2$  is uniformly continuous on  $[0, a]$ , where  $a > 0$ .
- (e) Show that  $\lim_{x \rightarrow 0} (x + \text{sgn}(x))$  does not exist.

3. Answer **any four** parts :  $5 \times 4 = 20$

(a) Evaluate —

$$\lim_{x \rightarrow 0^+} \left( \frac{1}{x} - \frac{1}{\sin x} \right)$$

(b) Let  $f: [a, b] \rightarrow \mathbb{R}$  be differentiable on  $[a, b]$ . If  $f'$  is positive on  $[a, b]$ , then prove that  $f$  is strictly increasing on  $[a, b]$ .

(c) Give an example to show that the product of two uniformly continuous function is not uniformly continuous on  $\mathbb{R}$ .

(d) Show that the function  $f(x) = |x|$  is continuous at every point  $c \in \mathbb{R}$ .

(e) Let  $f(x) = |2x|^{-\frac{1}{2}}$  for  $x \neq 0$ . Show that

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) = +\infty.$$

(f) If  $A \subseteq \mathbb{R}$  and  $f: A \rightarrow \mathbb{R}$  has a limit at  $c \in \mathbb{R}$ , then prove that  $f$  is bounded on some neighbourhood of  $c$ .

4. Answer **any four** parts :  $10 \times 4 = 40$

(a) State and prove Taylor's theorem.

(b) State and prove Mean value theorem. Give the geometrical interpretation of the theorem.  $(2+5)+3=10$

(c) (i) If  $f: I \rightarrow \mathbb{R}$  has derivative at  $c \in I$ , then prove that  $f$  is continuous at  $c$ . Is the converse true? Justify.

(ii) If  $r$  is a rational number, let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{for } x \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

Determine those values of  $r$  for which  $f'(0)$  exists.  $6$

(d) State and prove maximum-minimum theorem.  $2+8=10$

(e) (i) Let  $I$  be an interval and let  $f: I \rightarrow \mathbb{R}$  be continuous on  $I$ . Prove that the set  $f(I)$  is an interval.  $5$

(ii) Show that the function

$$f(x) = \frac{1}{1+x^2} \text{ for } x \in \mathbb{R} \text{ is}$$

uniformly continuous on  $\mathbb{R}$ . 5

(f) Prove that the rational functions and the sine functions are continuous on  $\mathbb{R}$ .

(g) (i) Give examples of functions  $f$  and  $g$  such that  $f$  and  $g$  do not have limits at a point  $c$  but such that both  $f+g$  and  $fg$  have limits at  $c$ . 6

(ii) Let  $A \subseteq \mathbb{R}$ , let  $f: A \rightarrow \mathbb{R}$  and let  $c$  be a cluster point of  $A$ . If  $\lim_{x \rightarrow c} f(x)$  exists and if  $|f|$  denotes the function defined for  $x \in A$  by  $|f|(x) = |f(x)|$ , Prove that

$$\lim_{x \rightarrow c} |f|(x) = \left| \lim_{x \rightarrow c} f(x) \right| \quad 4$$

(h) Let  $f: A \rightarrow \mathbb{R}$  and let  $c$  be a cluster point of  $A$ . Prove that the following are equivalent.

(i)  $\lim_{x \rightarrow c} f(x) = l$

(ii) For every sequence  $(x_n)$  in  $A$  that converges to  $c$  such that  $x_n \neq c$  for all  $n \in \mathbb{N}$ , the sequence  $(f(x_n))$  converges to  $l$ .