

*Depth -*  
Total number of printed pages-11

3 (Sem-1/CBCS) MAT HC 2

2025

**MATHEMATICS**

(Honours)

Paper : MAT-HC-1026

(Algebra)

Full Marks : 80

Time : Three hours

***The figures in the margin indicate full marks for the questions.***

1. Answer the following questions :  $1 \times 10 = 10$

(a) Fill in the blanks :

If  $A$  is a triangular matrix then  $\det A$  is the product of the entries on the \_\_\_\_\_.

(b) State 'true' or 'false' with justification :

Each column of  $AB$  ( where  $A$  and  $B$  are matrices whose product  $AB$  is defined) is a linear combination of the columns of  $B$  using weights from the corresponding columns of  $A$ .

(c) State 'true' or 'false' with justification :

If one row in an echelon form of an augmented matrix is  $[0 \ 0 \ 0 \ 5 \ 0]$ , then the associated linear system is inconsistent.

(d) State the well-ordering principle in  $\mathbb{N}$ .

(e) Let  $M(2, \mathbb{R})$  denote the set of all  $2 \times 2$  matrices over  $\mathbb{R}$ .

Consider the function  $f : M(2, \mathbb{R}) \rightarrow \mathbb{R}$  given by  $f(A) = \det A$ .

Show that  $f$  is not one-one.

(f) If  $A$  is the set of all  $n \times n$  symmetric matrices and  $B$  is the set of all  $n \times n$  real skew-symmetric matrices, what is  $A \cap B$  ?

(g) Write the statement and its negation using quantifiers.

"In each tree in the garden, we can find a branch on which all of the leaves are green."

(h) If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is given by  $f(x) = x^2$ , what is  $f^{-1}((0, 4))$  ?

(i) Find the Cartesian co-ordinates of the point  $(2, \frac{2\pi}{3})$ .

(j) Find the polar representation of the point  $(2, -2)$ .

2. Answer the following questions :  $2 \times 5 = 10$

(a)  $A$  is  $3 \times 3$  matrix with three pivot positions. Explain the following —

(i) Does  $A\bar{x} = \bar{0}$  have a nontrivial solution ?

(ii) Does  $A\bar{x} = \bar{b}$  have at least one solution for all  $\bar{b}$  in  $\mathbb{R}^3$  ?

(b) Find standard matrix of  $T$ , where  $T$  is a linear transformation such that  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  rotates points about the origin through  $\frac{3\pi}{2}$  in a counter-clockwise manner.

(c) Consider the relation on  $\mathbb{R}$  defined by  $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x \leq y\}$ . Is this relation an equivalence relation? Justify.

(d) Describe the following set explicitly and mark it on the real line

$$X = \{x \in \mathbb{R} \mid x(x-1)(x-2) < 0\}$$

(e) Compute  $(1+i)^{100}$ .

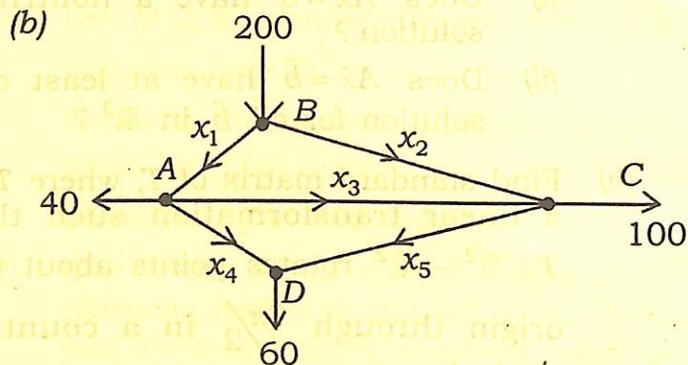
3. Answer **any four** questions :  $5 \times 4 = 20$

(a) Use Cramer's Rule to compute the solutions of the system

$$2x_1 + x_2 + x_3 = 4$$

$$-x_1 + \quad + 2x_3 = 2$$

$$3x_1 + x_2 + 3x_3 = -2$$



For the figure above find the general traffic pattern in the freeway network (Flow rates are cars/minute).

Describe the general traffic pattern when the road whose flow is  $x_4$  is closed.

(c) Describe the solutions of the following system in parametric form.

$$x_1 + 3x_2 - 5x_3 = 4$$

$$x_1 + 4x_2 - 8x_3 = 7$$

$$-3x_1 - 7x_2 + 9x_3 = -6$$

(d) For any sets  $A$  and  $B$ , show that the following are equivalent.

(i)  $A \subseteq B$

(ii)  $A \cup B = B$

(iii)  $A \cap B = A$

(iv)  $B^c \subseteq A^c$

(e) Let  $X = \mathbb{R} = Y$ . Let  $A = \{1\}$  and  $B = \mathbb{R}$ . Draw the sketch of  $A \times B$  as a subset of  $\mathbb{R}^2$ .

For  $A \subseteq X$  and  $B \subseteq Y$  show that there may be subsets of  $X \times Y$  that are not of the form  $A \times B$ .  $2+3=5$

(f) If  $n|q$  then prove that any root of  $z^n - 1 = 0$  is a root of  $z^q - 1 = 0$ . Prove that the common roots of  $z^m - 1 = 0$  and  $z^n - 1 = 0$  are roots of  $z^d - 1 = 0$  where  $d = \text{g.c.d}(m, n)$

i.e.  $U_m \cap U_n = U_d$ .  $1+4=5$

4. Answer **any four** of the following:  
 $10 \times 4 = 40$

- (a) (i) Compute the determinant by co-factor expansion choosing the row or column that involves least amount of computation.

$$\begin{vmatrix} 6 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 2 & 0 & 0 & 0 \\ 8 & 3 & 1 & 8 \end{vmatrix} \quad 2$$

- (ii) State 'true' or 'false' with justification:

The  $(i, j)^{th}$  cofactor of a matrix  $A$  is the matrix  $A_{ij}$  obtained from  $A$  by deleting the  $i^{th}$  row and  $j^{th}$  column of  $A$ . 2

- (iii) For the matrix given below-

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix}$$

- (a) compute the determinant.  
 (b) what is determinant of an elementary row replacement of the matrix?

- (c) what is the determinant of an elementary scaling matrix with  $k$  on the diagonal? 3

- (iv) Use a determinant to decide if  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  are linearly independent, where

$$\vec{v}_1 = \begin{bmatrix} 5 \\ -7 \\ 9 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} -3 \\ 3 \\ 5 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 2 \\ -7 \\ 5 \end{bmatrix} \quad 3$$

- (b) (i) Write the following system in matrix form and use the inverse of the co-efficient matrix to solve

$$\begin{aligned} 3x_1 + 4x_2 &= 3 \\ 5x_1 + 6x_2 &= 7. \end{aligned} \quad 2$$

- (ii) Let  $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a linear transformation and let  $A$  be the standard matrix for  $T$ . Then show that  $T$  is invertible if and only if  $A$  is an invertible matrix. Show that the linear transformation  $S$  given by  $S(\vec{x}) = A^{-1}\vec{x}$  where  $S: \mathbb{R}^n \rightarrow \mathbb{R}^n$ , is the unique inverse of  $T$ . 4

- (iii) Find the inverse of the following matrix if it exists by performing suitable row operations on the augmented matrix

$$\left[ \begin{array}{ccc|c} 1 & 0 & -2 & 4 \\ -3 & 1 & 4 & \\ 2 & -3 & 4 & \end{array} \right]$$

- (c) (i) If  $A$  is a  $m \times n$  matrix and  $\vec{U}, \vec{V}$  are vectors in  $\mathbb{R}^n$ ,  $C$  is a scalar then prove that

$$A(\vec{U} + \vec{V}) = A\vec{U} + A\vec{V}$$

$$\text{and } A(C\vec{U}) = C(A\vec{U}) \quad 4$$

- (ii) If  $A$  is a square  $n \times n$  matrix then prove that the following statements are logically equivalent :

- $A$  is an invertible matrix
- There is an  $n \times n$  matrix  $C$  such that  $CA = I$
- The equation  $A\vec{x} = \vec{0}$  has only the trivial solution
- $A$  has  $n$  pivot positions
- $A$  is row equivalent to the  $n \times n$  identity matrix 6

- (d) (i) Prove the statement using contrapositive.

“For integers  $x, y$  if  $x+y$  is even, then  $x$  and  $y$  are both odd or both even.” 2

- (ii) Determine  $h$  such that the following matrix is the augmented matrix of a consistent linear system

$$\left[ \begin{array}{ccc|c} 2 & 8 & h & \\ 4 & 6 & 7 & \end{array} \right] \quad 2$$

- (iii) Determine if  $\vec{b}$  is a linear combination of the vectors formed from the column of  $A$ . 4

$$A = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 5 \\ -2 & 8 & -4 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 3 \\ -7 \\ -3 \end{bmatrix}$$

- (iv) Show that the set of two vectors  $\{\vec{v}_1, \vec{v}_2\}$  is linearly dependent if at least one of the vectors is a multiple of the other. 2

(e) Prove that if  $a, b$  are integers not both zero and  $d$  be the greatest common divisor of  $a$  and  $b$  then  $\exists$  integers  $x, y$  s.t.  $d = ax + by$ . Further, prove that two integers  $m$  and  $n$  are relatively prime if and only if  $\exists$  integers  $p$  and  $q$  s.t.  $pm + qn = 1$ . 6+4

(f) (i) Show that a function  $f : X \rightarrow Y$  is one-one if and only if  $f(A \cap B) = f(A) \cap f(B)$  holds for all subsets  $A$  and  $B$  of  $X$ . 6

(ii) Give an example of a relation that is reflexive and transitive but not symmetric. 2

(iii) Produce a counter example to disapprove the statement —  
 "For integers  $a, b, c$  if  $a$  divides  $bc$ , then  $a$  divides  $b$  or  $a$  divides  $c$ ."  
2

(g) (i) Show that the function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by  $f(x, y) = (ax + by, cx + dy)$  is a bijection if  $ad - bc \neq 0$ . Find the inverse of  $f$ . 5

(ii) For any sets  $A, B, C$  prove that  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ . 5

(h) (i) Compute  $z^n + \frac{1}{z^n}$  if  $z + \frac{1}{z} = \sqrt{3}$ . 5

(ii) Prove that 5  
 $\sin 5t = 16 \sin^5 t - 20 \sin^3 t + 5 \sin t$   
 $\cos 5t = 16 \cos^5 t - 20 \cos^3 t + 5 \cos t$