2025

MATHEMATICS

(Major)

Paper : MAT0500104

(Multivariate Calculus)

Full Marks : 60

Time : 2½ hours

The figures in the margin indicate full marks for the questions

1. Answer the following as directed : 1×8=8

(a) If

$$f(x, y) = \frac{1}{\sqrt{9 - x^2 - y^2}}$$

then find the domain of it.

(b) Define level curve of a function $f(x, y)$ at a constant c.

(c) Find f_x if $f(x, y) = x^2 \sin(3x + y^3)$.

(d) If $f(x, y) = \tan xy$, then df is

(i) $y\sec^2 xy\,dy + x\sec^2 xy\,dx$

(ii) $y\sec^2 x\,dy\,dx + x\sec^2 xy\,dy$

(iii) $\sec^2 xy\,dx + \sec^2 xy\,dy$

(iv) $\sec^2 x\,dy + \sec^2 y\,dx$

(Choose the correct answer)

(e) If $P_0(x_0, y_0)$ is a critical point of $f(x, y)$ and f has continuous 2nd-order partial derivatives in a disk centered at (x_0, y_0) and $D = f_{xx}f_{yy} - f_{xy}^2$, then a relative maximum occurs at P_0, if

(i) $D(x_0, y_0) > 0$ and $f_{yy}(x_0, y_0) > 0$

(ii) $D(x_0, y_0) > 0$ and $f_{yy}(x_0, y_0) < 0$

(iii) $D(x_0, y_0) < 0$ and $f_{yy}(x_0, y_0) > 0$

(iv) $D(x_0, y_0) < 0$ and $f_{yy}(x_0, y_0) < 0$

(Choose the correct answer)

(f) Find the Jacobian of the transformation from Cartesian coordinate to polar coordinate system.

(g) State when a vector field C is said to be conservative.

(h) Find curl of the vector field

$$\vec{V}(x, y, z) = u(x, y, z)\hat{i} + v(x, y, z)\hat{j} + w(x, y, z)\hat{k}$$

2. Answer any *six* of the following questions : 2×6=12

(a) Show that

$$\lim_{(x, y)\to(0, 0)} \frac{x^2 y}{x^4 + y^2}$$

does not exist.

(b) Compute the slope of the tangent line to the graph of $f(x, y) = x\ln(x + y^2)$ at the point $P_0(e, 0, -e)$ in the direction parallel to the XZ-plane.

(c) Find the critical point of

$$f(x, y) = (x-2)^2 + (y-3)^4$$

and classify them.

(d) Find $\iint_R x\sin xy\,dA$ where

$$R = \{(x, y) : 0 \le x \le \pi,\ 0 \le y \le 1\}$$

(e) If $\vec{F} = \vec{\nabla} f$ is the gradient of the function

$$f(x, y, z) = -\frac{1}{x^2 + y^2 + z^2}$$

then by using fundamental theorem of line integral, find the work done by $\vec{F}$ along the smooth curve C joining (1, 0, 0) to (0, 0, 2) that not passes through the origin.

(f) Evaluate $\int_4^7 \int_1^2 \int_0^3 x^2y^2z^2 \, dx\, dy\, dz$.

(g) Find a parametrization of the cylinder $x^2 + (y-3)^2 = 9$, $0 \le z \le 5$.

(h) Find $\operatorname{div}\vec{F}$, given that $\vec{F} = \nabla f$ where $f(x, y, z) = x^2yz^3$.

(i) State Green's theorem.

(j) Evaluate $\int_C (x+y)\, dx$ if C be the curve represented by $x = 2t$, $y = 3t^2$, $0 \le t \le 1$.

3. Answer any *four* of the following questions : 5×4=20

(a) Express $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in terms of r and s if $w = x + 2y + z^2$, $x = \frac{r}{s}$, $y = r^2 \ln s$, $z = 2r$.

(b) Show that

$$f(x, y) = \begin{cases} \frac{2xy}{x^2+y^2}, & (x, y) \ne (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is continuous at every point except at the origin.

(c) Find the equation of the tangent plane and the normal line at $P_0(1, -1, 2)$ on the surface S is given by $x^2y + y^2z + z^2x = 5$.

(d) Find all critical points on the graph $f(x, y) = 8x^3 - 24xy + y^3$ and use the second partial test to classify as a relative extrema or a saddle point.

(e) Find the work done by the force field $\vec{F} = (x^2 + y^2)\hat{i} + (x+y)\hat{j}$ as an object moves counter clockwise along the circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(-1, 0)$ and then back to $(1, 0)$ along the x-axis.

(f) Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where

$$\vec{F} = (y^2 - z^2)\hat{i} + 2yz\,\hat{j} - x^2\hat{k}$$

and C is curve defined parametrically by $x = t^2$, $y = 2t$, $z = t$ for $0 \le t \le 1$.

(g) Find the mass of a lamina of density $\delta(x, y, z) = z$ in the shape of the hemisphere $z = (a^2 - x^2 - y^2)^{\frac{1}{2}}$.

(h) Evaluate the integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx$$

4. Answer any *two* of the following questions : 10×2=20

(a) (i) Suppose the function f is differentiable at the point P_0 and that the gradient at P_0 satisfies $\vec{\nabla} f_0 \neq 0$. Then show that $\vec{\nabla} f$ is orthogonal to the level surface of f through P_0. 5

(ii) Let $f(x, y, z) = xyz$ and let $\vec{u}$ be a unit vector perpendicular to both $\vec{v} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{w} = \hat{i} + \hat{j} - \hat{k}$. Find the directional derivative of f at $P_0(1, -1, 2)$ in the direction of $\vec{u}$. 5

(b) Show that the vector field

$$\vec{F} = 2x(y^2 + z^2)\hat{i} + 2x^2y\hat{j} + 3x^2z^2\hat{k}$$

is conservative. Also find its scalar potentials. 4+6=10

(c) Evaluate

$$\iiint_D (x^2 + y^2 + z^2)\, dx\, dy\, dz$$

where D denotes the region bounded by $x = 0$, $y = 0$, $z = 0$ and $x + y + z = a$, $a > 0$. 10

(d) Use the method of Lagrange's multiplier to minimize

$$f(x, y) = 16 - x^2 - y^2$$

subject to $x + 2y = 6$. 10

(e) (i) State divergence theorem. By using divergence theorem, evaluate $\iint_S \vec{F} \cdot \vec{N}\, dS$ where

$$\vec{F} = x^2\hat{i} + xy\hat{j} + x^3y^3\hat{k}$$

and S is the surface of the tetrahedron bounded by the plane $x + y + z = 1$ and the coordinate planes with outward unit normal vector $\vec{N}$. 1+5=6

(ii) Find the area of the region D bounded above the line $y = x$ and below by the circle $x^2 + y^2 - 2y = 0$. 4

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