2025

MATHEMATICS

(Major)

Paper : MAT0500404

(Abstract Algebra)

Full Marks : 60

Time : 2½ hours

The figures in the margin indicate full marks for the questions.

1. Answer the following : 1×8=8

(a) State the Lagrange's theorem on order of subgroups of a group.

(b) If H and K are two finite subgroups of a group, then which of the following is true?

(i) $O(HK) = \dfrac{O(H)+O(K)}{O(H \cap K)}$

(ii) $O(HK) = \dfrac{O(H)\,O(K)}{O(H \cap K)}$

(iii) $O(HK) = O(H) + O(K)$

(iv) $O(HK) = O(H)\,O(K)$

(c) Write True or False :
"Order of a cyclic group is equal to the order of its generators."

(d) Give an example of a left ideal which is not a right ideal.

(e) Express the permutation

$$\begin{pmatrix} a & b & c & d & e & f & g \\ c & d & e & g & f & b & a \end{pmatrix}$$

as a cycle.

(f) Under what condition,

$$Z_p = \{0, 1, 2, \cdots, (p-1)\}$$

modulo p will be a field?

(g) When will an element in a ring be called a nilpotent element?

(h) Give an example of a prime ideal of a ring which is not a maximal ideal in that ring.

2. Answer any *six* from the following : 2×6=12

(a) Let G be a group and a be any element of G. Show that $< a >$, the subset generated by a, is a subgroup of G.

(b) Define a ring homomorphism and its kernel.

(c) Show that every subgroup of a cyclic group is a cyclic group.

(d) Define the centre of a group and calculate the centre of S_3.

(e) Show that the centre of any group G is a normal subgroup of G.

(f) If $G = Z(G)$ is cyclic, then show that G is Abelian.

(g) If Z is the ring of integers and $H = \{3n : n \in Z\}$, then write all right cosets of H in Z.

(h) If x is an element of a group and x^n = the identity element of the group, then show that the order of x divides n.

(i) Show that every ideal of the ring of integers is a principal ideal.

(j) If $f : R \to S$ is a ring homomorphism, then show that kernel of f is an ideal of R.

3. Answer any *four* from the following : 5×4=20

(a) If f is a group homomorphism from G onto H, then show that $H \cong G/K$, where K is the kernel of f.

(b) Describe in pictures the elements of the dihedral group D_4 of the symmetries of a square.

(c) Write all subgroups of Z_{30}.

(d) Suppose $f : G \to H$ is a group homomorphism, then prove the following : 2+2+1=5

(i) If e is the identity of G, then $f(e)$ is the identity of H.

(ii) If K is a subgroup of G, then $f(K)$ is a subgroup of H.

(iii) If a is an element of G, then $f(a^{-1}) = (f(a))^{-1}$.

(e) Define a transposition in permutation. Show that every permutation can be expressed as a product of transpositions.

(f) If D is a commutative ring without zero-divisors, show that the characteristic of D is either a zero or a prime number.

(g) If P and Q are two ideals of a ring R, then show that $P+Q$ is again an ideal of R containing each of P and Q.

(h) Define zero-divisor in a ring. Show that a finite integral domain is a field. Give an example of an integral domain which is not a field. 1+3+1=5

4. Answer any *two* from the following : 10×2=20

(a) *(i)* Define a quotient group. Show that every quotient group of a cyclic group is cyclic. 5

(ii) Show that a group of prime order has no non-trivial subgroup. 2

(iii) If a is an integer and p is a prime, show that $a^p \equiv a (\text{mod } p)$. 3

(b) *(i)* Define even and odd permutations. Show that a cycle of even length is an odd permutation and a cycle of odd length is an even permutation. 2+3=5

(ii) Compute $a^{-1}bab^{-1}$ where $a = (135)(14)$ and $b = (2576)$. 4

(iii) Find the generators of the group $\{1, -1, i, -i\}$ with multiplication. 1

(c) Show that every group is isomorphic to a permutation group.

(d) If R is a commutative ring with unity, then show that an ideal M will be a maximal ideal in R if and only if R / M is a field.

(e) *(i)* Prove that a group homomorphism f will be one-one if and only if the kernel of f contains only the identity element. 5

(ii) Prove that a subgroup of index 2 in a group is a normal subgroup. 5

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