ISSN 0019-5588

Volume 52 • Number 3 • September 2021

Indian Journal of Pure & Applied Mathematics

Official Publication of the
Indian National Science Academy

Springer

This condition is known as the *pseudo-inverse condition* on the basis **B**. A *table algebra* (TA) is an associative algebra with involution $*$ over $\mathbb{C}$ that has a $*$-invariant basis $\mathbf{B} = \{b_0 = 1, b_1, \ldots, b_{r-1}\}$ with nonnegative real structure constants that satisfies the pseudo-inverse condition. Table algebras (abbr. TAs) share many properties in common with adjacency algebras of ASs, this is primarily due to the existence of a *degree* map that plays the role of the valency/augmentation map. The existence of the degree map is a consequence of Perron-Frobenius theory (see [2]). If the structure constants are only required to be real instead of nonnegative real in the TA definition, one obtains Blau's definition of *reality-based algebras* (see [9]). Algebraically there is a big price for doing this, because one may no longer have a map to play the role of the augmentation/valency map. For a generalization in the direction of CAs, one can replace the condition that the distinguished basis **B** contains 1 with the existence of a set of fibers $\Delta \subset \mathbf{B}$ that sums to 1, and suitably modify the pseudo-inverse condition to: $\forall i, \exists! j, k$ such that $b_k \in \Delta$ and $\lambda_{ijk} = \lambda_{jik} > 0$. This gives Lusztig's definition of *based algebras* [41].

The existence of a degree map implies that any TA basis has a unique rescaling by positive scalars so that $\delta(b_i) = \lambda_{ii^*0}$ for all i. This rescaled basis is called the *standard* basis of the TA. Alternatively, one could rescale to obtain a *transitional* basis, which has the property that $\delta(b_i) = 1$ for alll i. When the transitional basis of a TA has integral structure constants, the free $\mathbb{Z}$-module of this basis produces a *fusion ring*. (Fusion rings are the finite rank versions of *fusion categories*, a topic currently receiving a lot of attention in representation theory - see [36].)

(1.2.3) The AS feasibility problem for SITAs. If the structure constants relative to the standard basis of a TA are all nonnegative integers, we refer to the TA as a *standard integral table algebra* (SITA). Adjacency algebras of ASs are special cases of SITAs. For these modifications of the CA or Adjacency algebra of AS definitions, integrality of structure constants is a strong restriction. An even stronger condition is the feasibility of a SITA or based algebra as the adjacency algebra of an AS or CC.

Question 1 (Feasibility Problem) *Which SITAs are represented by ASs? That is, if there exists a SITA* $\mathbf{B} = \{b_0 = 1, b_1, \ldots, b_{r-1}\}$ *with a given collection of nonnegative integer structure constants, is there a transpose-closed set of* $n \times n$ *01-matrices S that sums to* J_n *and is a basis of an* r*-dimensional subalgebra of* $M_n(\mathbb{C})$ *with the same structure constants?*

The general feasibility problem would be to find necessary and sufficient conditions for a table algebra basis **B** to be feasible as an AS. Bangteng Xu, using ideas concerning hypergroup actions, showed the feasibility condition is equivalent to the condition that the SITA $(A, \mathbf{B})$ act "maximally and irreducibly" on a set of order n. [53]. While checking this condition is computationally difficult, there are some practical necessary conditions:

- The multiplicity of each irreducible character of $\mathbb{C}\mathbf{B}$ in the standard feasible trace of the SITA must be positive integers. This is equivalent to the existence of an n-dimensional standard representation Φ of the table algebra, and reduces the problem to realizing the elements of $\Phi(\mathbf{B})$ as a set of transposed-closed 01-matrices that sum to 1.
- The structure constants must satisfy the "handshaking lemma": if $b_j^* = b_j$, then for all $b_i \in \mathbf{B}$ with $\delta(b_i) > 1$, $\lambda_i ji\delta(b_i)$ must be even (for a proof see [31, Lemma 7]).
- The spectrum of each $\Phi(b_i)$ must satisfy all necessary restrictions to be the spectrum of a graph - this is a mainstream topic for researchers in algebraic graph theory. (For the latest computational support for this problem see [48].)

Here are two instances of the feasibility problem that are more likely to be accessible by current methods:

Question 2 (a) *Is there a symmetric AS of rank 3 and order 65 with* $b_1^2 = 32b_0 + 15b_1 + 16b_2$*?*

(This is the smallest SRG not known to exist, there are many others, see [12] for an extensive list of symmetric P-polynomial SITAs defined by their intersection arrays, that are not known to exist. The missing Moore graph (a 57-regular SRG on 3250 vertices) is the most famous open case of this sort of feasibility problem.)

(b) [29] *Is there a noncommutative AS of rank 7 with exactly four asymmetric relations? (The construction of SITAs of this type in* [29] *does not produce ASs.)*

(1.3) Classifications of CCs and ASs. An effort to classify CCs and ASs using computers has been ongoing since the 1990s. Here we give the current state of these classification efforts:

- CCs have been classified using a computer up to order 15 by Ziv-Av [38]. In this paper Klin and Ziv-Av also describe the smallest CC, with order 14, that is not a two-orbit configuration.
- Hanaki and Miyamoto classified all ASs of orders $n \leq 30$, and $n = 32, 33$, and 34 [27], and gave partial classifications for many other orders. Their work reduced the classification of ASs of order 31 to the classification of DRTs of order 31, which was just completed in November 2019 by Hanaki, Kharaghani, Mohammadian, and

Tayfeh-Rezaie (see [26]). For order 35, DRTs and rank 4 ASs remain unclassified to date. The smallest non-Schurian AS corresponds to a DRT of order 15.

SITAs of ranks 3 and 4 have been classified, but due to the feasibility problem it is hard to say which ones correspond to ASs. Rank 3 SITAs are classified by the above parameter restrictions on SRGs and DRTs. A similar classification of parameters can be accomplished for symmetric or asymmetric SITAs of rank 4. For ranks 5 and higher, certain types of ASs can be ruled out by character theory restrictions:

- Noncommutative rank 5 ASs do not exist [55] - neither do noncommutative rank 5 SITAs [32].
- There are no noncommutative rank 7 ASs with 6 asymmetric relations [29]. All examples known to the authour have 2 asymmetric relations, this is the inspiration for Question 2b above.

(1.4) And then comes Terwilliger algebras... In 1992 Paul Terwilliger published the first of a series of three papers on the subconstituent algebra of a CC. In the case of commutative ASs, this algebra extends the adjacency algebra of the AS by a copy of its dual algebra.

Definition 3 [47] Let $S = \{A_0 = I, A_1, \ldots, A_{r-1}\}$ be set of adjacency matrices for a CC (or AS) of rank r and order n.

Fix a vertex x in the AS S (i.e. the index of a designated row). For each A_i, let $E_i^* = E_i^*(x)$ be the diagonal idempotent matrix whose yy-entry is 1 if $(A_i)_{xy} = 1$, and otherwise 0. If $E^*(x) = \{E_i^*(x) : i = 0, 1, \ldots, r-1\}$ is this dual idempotent basis then $\mathbb{Q}E^*(x)$ is an r-dimensional semisimple commutative $\mathbb{Q}$-algebra.

The *rational Terwilliger (or subconstituent) algebra at vertex* x is the $\mathbb{Q}$-subalgebra of $M_n(\mathbb{Q})$ generated by the set of 01-matrices $S \cup E^*(x)$.

At this point we invite the reader to compute that for $S = K_n$ and each choice of x, $T_x(S)$ is a 5-dimensional coherent algebra. In general, there is one $T_x(S)$ up to isomorphism for each orbit of the combinatorial automorphism group $Aut(S)$. When x and y are not in the same orbit of $Aut(S)$, it may be the case that $T_x(S)$ and $T_y(S)$ are not isomorphic, and can even have different dimension. T_x is a semisimple subalgebra of a coherent algebra (its coherent closure), but it is often the case that it is not coherent.

Terwilliger algebras have been an area of intense study for SRGs and DRGs since they were first introduced (see the survey [43]). Deep connections to orthogonal polynomials were soon realized as generators of the Terwilliger algebras were found to satisfy the same identities as the universal algebras of the quantum groups related to certain families of orthogonal polynomials. This led to a characterization of irreducible representations in many of these casee, and to interesting connections of ASs to the spin models of conformal field theory (see Bannai's recent survey [5].) The solution of a rationality problem for Terwilliger algebras was a key step in the recent solution of the Bannai-Ito conjecture by Bang, Dubickas, Koolen, and Moulton showing that there are a finite number of association schemes of rank > 3 that are both metric and cometric (i.e. the dual is also metric) [3].

The rational Terwilliger algebra for a thin AS (i.e. a finite group G) of order n will be isomorphic to the full matrix algebra $M_n(\mathbb{Q})$. For any other AS, the dimension of the Terwilliger algebra $T_x(S)$ exceeds the rank of S for every x. For SRGs (symmetric ASs of rank 3), Yamazaki and Tomiyama proved

Theorem 4 [50] *For a symmetric rank 3 AS (i.e.* b_1 *is a SRG), and any vertex* x*,*

$$\mathbb{C} \otimes T_x(S) \simeq M_3(\mathbb{C}) \oplus M_2(\mathbb{C})^{m_2(x)} \oplus \mathbb{C}^{m_1(x)},$$

where $m_1(x)$ *and* $m_2(x)$ *can be determined from certain eigenspaces of a principal submatrix of* b_1*.*

The Terwilliger algebra of an AS of rank r always has one simple component isomorphic to $M_r(\mathbb{Q})$. This is the component corresponding to the *principal* irreducible $T_x(S)$-module generated by the elementary basis vector e_x [47]. It is easy to see that this module restricts to the regular module for both $\mathbb{Q}S$ and the r-dimensional diagonal commutative algebra $\mathbb{Q}[E^*(x)]$. The nonprincipal irreducible $T_x(S)$-modules are much more mysterious. Though it is rare, these can have dimension equal or exceeding the rank of S.

On the other hand, very little work has been done on Terwilliger algebras for general ASs and CCs, other than the very recent work of Muzychuk and Xu giving a formula for the Terwilliger algebra $T_x(S wr U)$ of the wreath product of two CCs S and U in terms of $T_x(S)$ and $T_x(U)$. For the Terwilliger algebras of DRTs, the author has recently asked:

Question 3 *Suppose S is an asymmetric AS of rank 3 and order* $4u + 3$, $u \geq 0$*, (i.e. a DRT), and x is a vertex of S.*

- *Is* dim $T_x(S)$ *always odd and* $\leq 8u + 9$*?*
- *Is the principal irreducible* $T_x(S)$*-module the unique one of maximal dimension 3?*