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3 (Sem–5/CBCS) MAT HC 1 (N/O)

2022

MATHEMATICS

(Honours)

Paper : MAT–HC–5016

(For New Syllabus)

(Complex Analysis)

Full Marks : 60

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer ***any seven*** questions from the following : $1 \times 7 = 7$

(a) Describe the domain of definition of the function $f(z) = \dfrac{z}{z+\overline{z}}$.

(b) What is the multiplicative inverse of a non-zero complex number $z = (x, y)$?

Contd.

(c) Verify that $(3, 1)\ (3, -1)\ \left(\frac{1}{5}, \frac{1}{10}\right) = (2, 1)$.

(d) Determine the accumulation points of the set $Z_n = \frac{i}{n}$ $(n = 1, 2, 3, \ldots)$.

(e) Write the Cauchy-Riemann equations for a function $f(z) = u + iv$.

(f) When a function f is said to be analytic at a point?

(g) Determine the singular points of the function $f(z) = \frac{2z+1}{z(z^2+1)}$.

(h) $exp(2 \pm 3\pi i)$ is

(i) $-e^2$

(ii) e^2

(iii) $2e$

(iv) $-2e$ (Choose the correct answer)

(i) The value of $\log(-1)$ is

(i) 0

(ii) $2n\pi i$

(iii) πi

(iv) $-\pi i$ (Choose the correct answer)

(j) If $z = x + iy$, then $\sin z$ is

(i) $\sin x \cosh y + i \cos x \sinh y$

(ii) $\cos x \cosh y - i \sin x \sinh y$

(iii) $\cos x \sinh y + i \sin x \cosh y$

(iv) $\sin x \sinh y - i \cos x \cosh y$

(Choose the correct answer)

(k) If $\cos z = 0$, then

(i) $z = n\pi, (n = 0, \pm 1, \pm 2, \ldots)$

(ii) $z = \frac{\pi}{2} + n\pi, (n = 0, \pm 1, \pm 2, \ldots)$

(iii) $z = 2n\pi, (n = 0, \pm 1, \pm 2, \ldots)$

(iv) $z = \frac{\pi}{2} + 2n\pi, (n = 0, \pm 1, \pm 2, \ldots)$

(Choose the correct answer)

(l) If z_0 is a point in the z-plane, then $\lim_{z\to\infty} f(z) = \infty$ if

(i) $\lim_{z\to 0} \frac{1}{f(z)} = \infty$

(ii) $\lim_{z\to 0} f\left(\frac{1}{z}\right) = 0$

(iii) $\lim_{z\to 0} \frac{1}{f(z)} = 0$

(iv) $\lim_{z\to 0} \frac{1}{f\left(\frac{1}{z}\right)} = 0$

(Choose the correct answer)

2. Answer ***any four*** questions from the following : 2×4=8

(a) Reduce the quantity $\frac{5i}{(1-i)(2-i)(3-i)}$ to a real number.

(b) Define a connected set and give *one* example.

(c) Find all values of z such that $exp\,(2z-1) = 1$.

(d) Show that $log\left(i^3\right) \neq 3\,log\,i$.

(e) Show that

$$2\sin(z_1+z_2)\sin(z_1-z_2) = \cos 2z_2 - \cos 2z_1$$

(f) If z_0 and w_0 are points in the z plane and w plane respectively, then prove that $\lim_{z\to z_0} f(z) = \infty$ if and only if

$$\lim_{z\to z_0} \frac{1}{f(z)} = 0.$$

(g) State the Cauchy integral formula. Find $\frac{1}{2\pi i}\int_C \frac{1}{z-z_0}\,dz$ if z_0 is any point interior to simple closed contour C.

(h) Show that $\int_0^{\frac{\pi}{6}} e^{i2t}\,dt = \frac{\sqrt{3}}{4} + \frac{i}{4}$.

Contd.

3. Answer ***any three*** questions from the following : 5×3=15

(a) (i) If a and b are complex constants, use definition of limit to show that

$$\lim_{z \to z_0} (az+b) = az_0 + b.$$ 2

(ii) Show that

$$\lim_{z \to 0} \left(\frac{z}{\bar{z}}\right)^2 \text{ does not exist.}$$ 3

(b) Suppose that $\lim_{z \to z_0} f(z) = w_0$ and $\lim_{z \to z_0} F(z) = W_0$.

Prove that $\lim_{z \to z_0} [f(z)F(z)] = w_0 W_0$.

(c) (i) Show that for the function $f(z) = \bar{z}$, $f'(z)$ does not exist anywhere. 3

(ii) Show that $\lim_{z \to \infty} \frac{4z^2}{(z-1)^2} = 4$. 2

(d) (i) Show that the function $f(z) = exp\,\bar{z}$ is not analytic anywhere. 3

(ii) Find all roots of the equation

$$\log z = i\frac{\pi}{2}.$$ 2

(e) If a function f is analytic at all points interior to and on a simple closed contour C, then prove that

$$\int_C f(z)dz = 0.$$

(f) Evaluate : 2½+2½=5

(i) $\int_C \frac{e^{-z}}{z-(\pi i/2)} dz$

(ii) $\int_C \frac{z}{2z+1} dz$

where C denotes the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$.

(g) Prove that any polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n \; (a_n \neq 0)$$

of degree $n (n \geq 1)$ has at least one zero.

(h) Find the Laurent series that represents the function $f(z) = z^2 \sin\left(\frac{1}{z^2}\right)$ in the domain $0 < |z| < \infty$.

4. Answer ***any three*** questions from the following : 10×3=30

(a) (i) If a function f is continuous throughout a region R that is both closed and bounded, then prove that there exists a non-negative real number μ such that $|f(z)| \leq \mu$ for all points z in R, where equality holds for at least one such z. 4

(ii) Let a function $f(z) = u(x, y) + iv(x, y)$ be analytic throughout a given domain D. If $|f(z)|$ is constant throughout D, then prove that $f(z)$ must be constant there too. 3

(iii) Show that the function $f(z) = \sin x \cosh y + i \cos x \sinh y$ is entire. 3

(b) (i) Suppose that $f(z_0) = g(z_0) = 0$ and that $f'(z_0)$ $g'(z_0)$ exist, where $g'(z_0) \neq 0$. Use definition of derivative to show that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$ 3

(ii) Show that $f'(z)$ does not exist at any point if $f(z) = 2x + ixy^2$. 3

(iii) If a function f is analytic at a given point, then prove that its derivatives of all orders are analytic there too. 4

(c) Let the function $f(z) = u(x, y) + iv(x, y)$ be defined throughout some ε-neighbourhood of a point $z_0 = x_0 + iy_0$. If u_x, u_y, v_x, v_y exist everywhere in the neighbourhood, and these partial derivatives are continuous at (x_0, y_0) and satisfy the Cauchy-Riemann equations at (x_0, y_0), then prove that $f'(z_0)$ exist and $f'(z_0) = u_x + iv_x$ where the right hand side is to be evaluated at (x_0, y_0).

Use it to show that for the function $f(z) = e^{-x} . e^{-y}$, $f''(z)$ exists everywhere and $f''(z) = f(z)$. 6+4=10

(d) (i) Prove that the existence of the derivative of a function at a point implies the continuity of the function at that point.

With the help of an example show that the continuity of a function at a point does not imply the existence of derivative there. 3+5=8

(ii) Find $f'(z)$ if

$$f(z) = \frac{z-1}{2z+1} \left(z \neq -\frac{1}{2} \right).$$ 2

(e) (i) Prove that $\int_C \frac{dz}{z} = \pi i$ where C is the right-hand half $z = 2e^{i\theta}$ $\left(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \right)$ of the circle $|z| = 2$ from $z = -2i$ to $z = 2i$. 5

(ii) If a function f is analytic everywhere inside and on a simple closed contour C, taken in the positive sense, then prove that

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-z)^2} ds$$ where s denotes points on C and z is interior to C. 5

(f) (i) Evaluate $I = \int_C z^{a-1}\, dz$

where C is the positively oriented circle $z = Re^{i\theta}\ (-\pi \le \theta \le \pi)$ about the origin and a denote any non-zero real number.

If a is a non-zero integer n, then what is the value of $\int_C z^{n-1}\, dz$?

4+1=5

(ii) Let C denote a contour of length L, and suppose that a function $f(z)$ is piecewise continuous on C. If μ is a non-negative constant such that $|f(z)| \le \mu$ for all point z on C at which $f(z)$ is defined, then prove that $\left| \int_C f(z)\, dz \right| \le \mu L$.

Use it to show that $\left| \int_C \frac{dz}{z^2 - 1} \right| \le \frac{\pi}{3}$

where C is the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the Ist quadrant. 3+2=5

(g) (i) Apply the Cauchy-Goursat theorem to show that $\int_C f(z) = 0$ when the contour C is the unit circle $|z| = 1$, in either direction and $f(z) = ze^{-z}$. 4

(ii) If C is the positively oriented unit circle $|z| = 1$ and $f(z) = exp(2z)$ find $\int_C \frac{f(z)}{z^4} dz$. 3

(iii) Let z_0 be any point interior to a positively oriented simple closed curve C. Show that

$$\int_C \frac{dz}{(z - z_0)^{n+1}} = 0,\ (n = 1, 2, \ldots).$$ 3

(h) (i) Suppose that $z_n = x_n + iy_n$, $(n = 1, 2, \ldots)$ and $z = x + iy$. Prove that $\lim_{n\to\infty} z_n = z$ if and only if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. 5

(ii) Show that

$$z^2 e^{3z} = \sum_{n=2}^{\infty} \frac{3^{n-2}}{(n-2)!} z^n \ (|z| < \infty)$$

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(For Old Syllabus)

(Riemann Integration and Metric Spaces)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions : $1\times10=10$

   (a) Describe an open ball on the real line $\mathbb{R}$ for the usual metric d.

   (b) Find the limit point of the set $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$.

   (c) Define Cauchy sequence in a metric space (X, d).

   (d) Let A and B be two subsets of a metric space (X, d). Then

   (i) $(A \cap B)^0 = A^0 \cap B^0$

   (ii) $(A \cup B)^0 = A^0 \cup B^0$

   (iii) $(A \cap B)' = A' \cap B'$

   (iv) $(A \cup B)' = A' \cup B'$

   where A^0 denotes interior of A
   A' denotes derived set of A

   (Choose the correct answer)

   (e) In a complete metric space

   (i) every sequence is bounded

   (ii) every bounded sequence is convergent

   (iii) every convergent sequence is bounded

   (iv) every Cauchy sequence is convergent

   (Choose the correct answer)

   (f) Let $\{F_n\}$ be a decreasing sequence of closed subsets of a complete metric space and $d(F_n) \to 0$ as $n \to \infty$. Then

   (i) $\bigcap_{n=1}^{\infty} F_n = \phi$

   (ii) $\bigcap_{n=1}^{\infty} F_n$ contains at least one point

   (iii) $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point

   (iv) $d\left(\bigcap_{n=1}^{\infty} F_n\right) > 0$

   (Choose the correct answer)

(g) Let (X, d) and (Y, ρ) be metric spaces and $A \subset X$. Let $f : X \to Y$ be continuous on X. Then

(i) $f(A) = f(\overline{A})$

(ii) $f(\overline{A}) \subset \overline{f(A)}$

(iii) $\overline{f(A)} \subset f(\overline{A})$

(iv) $f(A) = f(A^0)$

(Choose the correct answer)

(h) What is meant by partition P of an interval $[a, b]$?

(i) Prove that $\lceil \alpha + 1 = \alpha \lceil \alpha$

(j) Define the upper and the lower Darboux sums of a function $f : [a, b] \to \mathbb{R}$ with respect to a partition P.

2. Answer the following questions : 2×5=10

(a) Prove that in a discrete metric space every singleton set is open.

(b) For any two subsets F_1 and F_2 of a metric space (X, d), prove that

$$\overline{(F_1 \cup F_2)} = \overline{F_1} \cup \overline{F_2}$$

(c) Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$. Then if f is continuous on X, prove that $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$ for all subsets B of Y.

(d) Find $L(f, P)$ and $U(f, P)$ for a constant function $f : [a, b] \to \mathbb{R}$.

(e) Examine the existence of improper integral $\int_0^1 \frac{1}{\sqrt{x}}\, dx$.

3. Answer **any four** parts : 5×4=20

(a) Let d be a metric on the non-empty set X. Prove that the function d' defined by

$$d'(x, y) = min\{1, d(x, y)\}$$

where $x, y \in X$ is a metric on X. State whether d' is bounded or not. 4+1=5

(b) In a metric space (X, d), prove that every closed sphere is a closed set.

(c) Prove that if a Cauchy sequence of points in a metric space (X, d) contains a convergent subsequence, then the sequence also converges to the same limit as the subsequence.

(d) Let (X, d) be a metric space and let $\{Y_\lambda, \lambda \in l\}$ be a family of connected sets in (X, d) having a non-empty intersection. Then prove that $Y = \bigcup_{\lambda \in l} Y_\lambda$ is connected.

(e) Consider the function $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = \begin{cases} 1 & \text{if } x \in Q \\ 0 & \text{otherwise} \end{cases}$

Prove that f is not integrable on [0, 1].

(f) Let $f : [a, b] \to R$ be bounded and monotone. Prove that f is integrable.

4. Answer ***any four*** parts : 10×4=40

(a) (i) Define a metric space.

Let

$$X = \mathbb{R}^n = \{x = (x_1, x_2, \dots x_n), x_i \in \mathbb{R}, 1 \le i \le n\}$$

be the set of all real n-tuples. For $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ in $\mathbb{R}^n$ define

$$d(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}.$$

Prove that $(\mathbb{R}^n, d)$ is a metric space. 2+4=6

(ii) Prove that in a metric space (X, d), a finite intersection of open sets is open. 4

(b) Let Y be a subspace of a metric space (X, d). Prove the following : 5+5=10

(i) Every subset of Y that is open in Y is also open in X if and only if Y is open in X.

(ii) Every subset of Y that is closed in Y is also closed in X if and only if Y is closed in X.

(c) (i) Prove that the function $f:[0,1]\to\mathbb{R}$ defined by $f(x)=x^2$ is uniformly continuous. Further prove that the function will not be uniformly continuous if the domain is $\mathbb{R}$. 3+3=6

(ii) Let (X, d_X), (Y, d_Y) and (Z, d_Z) be metric spaces and let $f: X\to Y$ and $g: Y\to Z$ be continuous. Prove that the composition $g\circ f$ is a continuous map of X into Z. 4

(d) When a metric space is said to be disconnected?

Prove that a metric space (X, d) is disconnected if and only if there exists a non-empty proper subset of X which is both open and closed in (X, d). 2+8=10

(e) (i) Show that the metric space (X, d) where X denotes the space of all sequences $x=(x_1, x_2, x_3, \ldots, x_n)$ of real numbers for which

$$\left(\sum_{k=1}^{\infty}|x_k|^p\right)^{\frac{1}{p}}<\infty\ (p\geq 1)$$ and d is the metric given by

$$d_p(x, y)=\left(\sum_{k=1}^{\infty}(x_k-y_k)^p\right)^{1/p},\quad x, y\in X$$

is a complete metric space. 7

(ii) Let X be any non-empty set and let d be defined by

$$d(x, y)=\begin{cases}0 & \text{if } x=y\\ 1 & \text{if } x\neq y\end{cases}$$

Show that (X, d) is a complete metric space. 3

(f) Prove that a bounded function $f:[a, b] \to R$ is integrable if and only if for each $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \varepsilon$.

(g) Let $f:[0, 1] \to \mathbb{R}$ be continuous. Let $C_i \in \left[\frac{i-1}{n}, \frac{i}{n}\right], n \in N$. Then prove that

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(C_i) = \int_0^1 f(x)\, dx.$$

Using it, prove that $\lim_{n \to \infty} \sum_{k=1}^{n} \frac{n}{k^2 + n^2} = \frac{\pi}{4}$.

6+4=10

(h) (i) Prove that a mapping $f: X \to Y$ is continuous on X if and only if $f^{-1}(F)$ is closed in X for all closed subsets F of Y. 5

(ii) Let f and g be continuous on $[a, b]$. Also assume that g does not change sign on $[a, b]$. Then prove that for some $c \in [a, b]$ we have

$$\int_a^b f(x)\, g(x)\, dx = f(c) \int_a^b g(x)\, dx.$$

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