*Total number of printed pages–8*

**3 (Sem–5 /CBCS) MAT HC 2**

**2021**

**( Held in 2022 )**

# MATHEMATICS

(Honours)

Paper : MAT–HC–5026

***( Linear Algebra )***

*Full Marks : 80*

Time : Three hours

***The figures in the margin indicate full marks for the questions.***

1. Answer the following as directed : 1×10=10

*(i)* Is $\mathbb{R}^2(\mathbb{R})$ is a subspace of $\mathbb{R}^3(\mathbb{R})$ ?

*(ii)* Let $A$ be a 5×4 matrix. If null space of $A$ is a subspace of $\mathbb{R}^k$ then what is $k$?

*(iii)* Let $S$ be a subset of a vector space $V(F)$ and $S$ contains zero vector of $V$. Then $S$ is

*(A)* linearly independent

*(B)* linearly dependent

*Contd.*

(C) Both linearly independent and linearly dependent

(D) None of the above

*(Choose the correct option)*

(iv) Write the standard basis of the vector space of polynomial in $x$ with real co-efficient of degree $\leq 3$.

(v) "The eigenvalues of a triangular matrix are the entries on its main diagonal." *(State True or False)*

(vi) Define inner product on $\mathbb{R}^n$.

(vii) Which vector is orthogonal to every vector in $\mathbb{R}^n$ ?

(viii) How do you explain $dim\, W = 1$ geometrically where $W$ is a subspace of the vector space $\mathbb{R}^3(\mathbb{R})$ ?



(ix) Let $A$ be the $4 \times 4$ real matrix,

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -2 & -2 & 0 \\ 1 & 1 & -2 & 1 \end{bmatrix}$$

Then the characteristic polynomial for $A$ is

(A) $x^2(x-1)^2$

(B) $(x-1)^2(x+1)^2$

(C) $x^2(x+1)^2$

(D) None of the above

*(Choose the correct option)*

(x) What do you mean by the length of a vector in $\mathbb{R}^n$ ?

2. Answer the following questions : $2 \times 5 = 10$

(i) Let $V$ be the vector space of all functions from the real field $\mathbb{R}$ to $\mathbb{R}$. Show that $W = \{f : f(7) = 2 + f(1)\}$ is not a subspace of $V$.

(ii) Show that every subset of an independent set is independent.

(iii) Let $A=\begin{bmatrix}1 & 6\\ 5 & 2\end{bmatrix}$ and $v=\begin{bmatrix}3\\ -2\end{bmatrix}$. Is $v$ a eigenvector of $A$ ?

(iv) Let $T$ be the linear operator on $\mathbb{R}^3$ defined by $T(a,b,c)=(a+b,\, b+c,\, 0)$. Show that the $xy$-plane $=\{(x,y,0): x,y\in\mathbb{R}\}$ is $T$-invariant subspace of $\mathbb{R}^3$.

(v) Let $y=\begin{bmatrix}7\\ 6\end{bmatrix}$ and $u=\begin{bmatrix}4\\ 2\end{bmatrix}$. Find the orthogonal projection of $y$ onto $u$.

3. Answer ***any four*** questions : 5×4=20

(i) Prove that the non-zero vectors $v_1, v_2, \ldots, v_n$ are linearly dependent if and only if one of them is a linear combination of the preceding vectors.

(ii) Let $v_1, v_2, \ldots, v_n$ be non-zero eigenvectors of an operator $T: V\to V$ corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Prove that $v_1, v_2, \ldots, v_n$ are linearly independent.



(iii) Let $A$ and $B$ be two similar matrices of order $n\times n$. Prove that $A$ and $B$ have same characteristic polynomial and hence the same eigenvalues.

(iv) Let $A=\begin{pmatrix}2 & 1 & 0\\ 0 & 1 & -1\\ 0 & 2 & 4\end{pmatrix}$. An eigenvalue of $A$ is 2. Find a basis for the corresponding eigenspace.

(v) Let $A=\begin{bmatrix}7 & 2\\ -4 & 1\end{bmatrix}$. Find a formula for $A^2$, given that $A=PDP^{-1}$ where $P=\begin{bmatrix}1 & 1\\ -1 & -2\end{bmatrix}$ and $D=\begin{bmatrix}5 & 0\\ 0 & 3\end{bmatrix}$.

(vi) Define orthogonal set. If $S=\{u_1, u_2, \ldots, u_p\}$ is an orthogonal set of non-zero vectors in $\mathbb{R}^n$, then prove that $S$ is linearly independent and hence is a basis for the subspace spanned by $S$.

4. *(i)* If a vector space $V$ has a basis $B=\{v_1,v_2,\ldots,v_n\}$, then prove that any set in $V$ containing more than $n$ vectors must be linearly dependent. Also show that every basis of $V$ must consist of exactly $n$ vectors. 5+5=10

**OR**

Let $U$ and $V$ be vector spaces over the same field. Let $\{u_1,u_2,\ldots,u_n\}$ be a basis of $U$ and let $v_1,v_2,\ldots,v_n$ be any arbitrary vectors in $V$. Prove that there exists a unique linear mapping $f:U\to V$ such that

$f(u_1)=v_1,\ f(u_2)=v_2,\ldots,f(u_n)=v_n$ 10

*(ii)* Find the eigenvalues and eigenvectors of $A=\begin{bmatrix}5 & 4\\ 1 & 2\end{bmatrix}$. 10

**OR**

State Cayley-Hamilton theorem for matrices. Use it to express $2A^5-3A^4-A^2-4I$ as a linear polynomial in $A$, when $A=\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}$. 10



*(iii)* Let $T$ be the linear operator on $\mathbb{R}^3$, defined by

$T(x,y,z)=(2y+z,\ x-4y,\ 3x)$

*(a)* Find the matrix of $T$ in the basis $\{e_1=(1,1,1),\ e_2=(1,1,0),\ e_3=(1,0,0)\}$

*(b)* Verify that $[T]_e[v]_e=[T(v)]_e$ for any vector $v\in\mathbb{R}^3$. 4+6=10

**OR**

An $n\times n$ matrix $A$ is diagonalizable if and only if $A$ has $n$ linearly independent eigen-vectors. 10

*(iv)* Define orthonormal set and orthonormal basis in $\mathbb{R}^n$. Show that $\{u_1,u_2,u_3\}$ is an orthonormal basis of $\mathbb{R}^3$, where

$$u_1=\begin{bmatrix}3/\sqrt{11}\\ 1/\sqrt{11}\\ 1/\sqrt{11}\end{bmatrix},\quad u_2=\begin{bmatrix}-1/\sqrt{6}\\ 2/\sqrt{6}\\ 1/\sqrt{6}\end{bmatrix},\quad u_3=\begin{bmatrix}-1/\sqrt{66}\\ -4/\sqrt{66}\\ 7/\sqrt{66}\end{bmatrix}$$

1+1+8=10

***OR***

Define inner product space. Show that the following is an inner product in $\mathbb{R}^2$:

$\langle u, v \rangle = x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2$

where $u = (x_1, x_2)$, $v = (y_1, y_2)$.

Also show that for all $u$, $v$ in $\mathbb{R}^2$

$\|u + v\| \leq \|u\| + \|v\|$ 2+5+3=10

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