

Total number of printed pages-8

3 (Sem-3/CBCS) MAT HC 2

2023

MATHEMATICS

(Honours Core)

Paper : MAT-HC-3026

(Group Theory-1)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions as directed :

$$1 \times 10 = 10$$

(a) Define order of an element of a group.

(b) In the group Q^* of all non-zero rational numbers under multiplication, list the

elements of $\left\langle \frac{1}{2} \right\rangle$.

(c) Find elements A, B, C in D_4 such that $AB = BC$ but $A \neq C$.

Contd.

- (d) Define simple group.
- (e) State Cauchy's theorem on finite Abelian group.
- (f) State whether the following statement is true **or** false:
 "If H is a subgroup of the group G and $a \in G$, then $Ha = \{ha : a \in G\}$ is also a subgroup of G ."
- (g) Write the order of the alternating group A_n of degree n .
- (h) Give an example of an onto group homomorphism which is not an isomorphism.
- (i) State whether the following statement is true **or** false :
 "If the homomorphic image of a group is Abelian then the group itself is Abelian."
- (j) Which of the following statement is true ?
- A homomorphism from a group to itself is called monomorphism.
 - A one-to-one homomorphism is called epimorphism.

- (c) An onto homomorphism is called endomorphism.
- (d) None of the above
2. Answer the following questions : $2 \times 5 = 10$
- In D_3 , find all elements X such that $X^3 = X$.
 - Consider the group Z_2 under $+_2$ and Z_3 under $+_3$. List the elements of $Z_2 \oplus Z_3$ and find $|Z_2 \oplus Z_3|$.
 - Express $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 1 & 4 & 3 & 2 \end{pmatrix}$ as product of transposition and find its order.
 - If $\psi : G \rightarrow G'$ is a group homomorphism and e and e' be the identity elements of the group G and G' respectively then show that $\psi(e) = e'$.
 - Show that in a group G , if the map $f : G \rightarrow G'$ defined by $f(x) = x^{-1}$, $\forall x \in G$ is a homomorphism then G is Abelian.

3. Answer **any four** questions : $5 \times 4 = 20$

(a) Let G be a group and H be a non-empty finite subset of G . Prove that H is a subgroup of G if and only if H is closed under the operation in G .

(b) If a is an element of order n in a group and k is a positive integer then prove that

$$\langle a^k \rangle = \langle a^{\gcd(n, k)} \rangle \text{ and}$$

$$|\langle a^k \rangle| = \frac{n}{\gcd(n, k)}.$$

(c) Show that a subgroup H of a group G is a normal subgroup of G if and only if product of two right cosets of H in G is again a right coset of H in G .

(d) If a, n are two integers such that $n \geq 1$ and $\gcd(a, n) = 1$, then prove that $a^{\phi(n)} \equiv 1 \pmod{n}$, where $\phi(n)$ is the Euler's phi function.

(e) Show that any finite cyclic group of order n is isomorphic to $\frac{\mathbb{Z}}{\langle n \rangle}$, where \mathbb{Z} is the additive group of integers and $\langle n \rangle = \{0, n, 2n, \dots\}$.

(f) Let $\sigma: G \rightarrow \bar{G}$ be a group homomorphism and $a, b \in G$.

(i) Show that

$$\sigma(a) = \sigma(b) \Leftrightarrow a \ker \sigma = b \ker \sigma.$$

(ii) If $\sigma(g) = g'$ then show that

$$\sigma(g') = \{x \in G : \sigma(x) = g'\} = g \ker \sigma.$$

$$2+3=5$$

Answer **either (a) or (b)** from the following questions : $10 \times 4 = 40$

4. (a) Describe the elements of D_4 , the symmetries of a square. Write down a complete Cayley's table for D_4 . Show that D_4 forms a group under composition of functions. Is D_4 an Abelian group? $2+3+4+1=10$

(b) Prove that every subgroup of a cyclic group is cyclic. Also show that if $|\langle a \rangle| = n$, then the order of any subgroup of $\langle a \rangle$ is a divisor of n .

Moreover, show that the group $\langle a \rangle$ has

exactly one subgroup $\langle a^{\frac{n}{k}} \rangle$ of order k .

Find the subgroup of Z_{30} which is of order 3. $4+2+3+1=10$

5. (a) Show that every quotient group of a cyclic group is cyclic. Give example to show that converse of this statement is

not true in general. Find $\frac{\mathbb{Z}}{N}$ where \mathbb{Z} is the additive group of integers and $N = \{5n : n \in \mathbb{Z}\}$. 4+3+3=10.

- (b) (i) Show that every finite group can be represented as a permutation group. 7

- (ii) Let $\phi : G \rightarrow \bar{G}$ be a group homomorphism and H be a subgroup of G . If \bar{K} is a normal subgroup of \bar{G} then show that $\phi^{-1}[\bar{K}] = \{k \in G : \phi(k) \in \bar{K}\}$ is a normal subgroup of G . 3

6. (a) (i) State and prove Lagrange's theorem for the order of subgroup of a finite group. Is the converse true? Justify your answer.

$$1+5+1=7$$

- (ii) List the elements of $\frac{\mathbb{Z}}{4\mathbb{Z}}$ and construct a Cayley's table for it. 3

- (b) (i) Show that any two disjoint cycles commute. 5

- (ii) Let G be a group and $Z(G)$ be the center of G . If $\frac{G}{Z(G)}$ is cyclic then show that G is Abelian. 5

7. (a) Let G be a group and H be any subgroup of G . If N is any normal subgroup of G , then show that :

- (i) $H \cap N$ is a normal subgroup of H .

- (ii) N is a normal subgroup of HN .

(iii) $\frac{HN}{N} \cong \frac{H}{H \cap N}$.

$$2+2+6=10$$

(b) Let $f : G \rightarrow G'$ be an onto group homomorphism and H be a subgroup of G , H' a subgroup of G' . Prove that :

(i) $f[H]$ is a subgroup of G' .

(ii) $f^{-1}[H']$ is a subgroup of G containing $K = \ker f$, where

$$f^{-1}[H'] = \{x \in G : f(x) \in H'\}.$$

(iii) There exists a one-to-one correspondence between the set of subgroups of G containing K and set of subgroups of G' .

$$2+3+5=10$$