2019

MATHEMATICS

(Major)

Paper : 5.2

(Topology)

Full Marks : 60

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer the following questions : $1\times7=7$

(a) Let $c[a, b]$ denote the set of all real-valued continuous functions defined on the interval $[a, b]$. Define a metric on $c[a, b]$ for which it is not complete.

(b) Describe open spheres of unit radius about the point (0, 0) for the following metric on $\mathbb{R}^2$:

$$d(z_1, z_2) = \max\{|x_1 - x_2|, |y_1 - y_2|\},$$

$$z_1 = (x_1, y_1), z_2 = (x_2, y_2) \in \mathbb{R}^2$$

(c) Give an example to show that the union of an infinite collection of closed sets in a metric space is not necessarily closed.

(d) What do you mean by metric topology? Give an example.

(e) Give an example to show that the union of two topologies need not be a topology.

(f) Let (X, D) be the indiscrete topological space. Find the closed subsets of X.

(g) What is a Hilbert space? Give one example.

2. Answer the following questions : 2×4=8

(a) Every subset of a discrete metric space is both open and closed. Justify whether it is true or false.

(b) Which of the following subsets of $\mathbb{R}$ are neighbourhoods of 1 with respect to the usual topology on $\mathbb{R}$?

(i) $]0, 2[$

(ii) $]0, 2]$

(iii) $[1, 2]$

(iv) $]1, 2]$

(v) $[1, 2[$

Justify your answer.

(c) If $(X, \|\cdot\|)$ is a normed linear space, then explain how a metric d can be defined on X using the norm $\|\cdot\|$.

(d) Every inner product space is a normed linear space. Justify whether it is true or false.

3. Answer the following questions : 5×3=15

(a) Let (X, d) be a metric space and $G \subset X$ be an arbitrary set. Show that G is open $\Leftrightarrow$ it is a union of open spheres.

(b) On the set of real numbers $\mathbb{R}$, let u consist of ϕ and all those subsets G of $\mathbb{R}$ having the property that to each $x \in G$, there exists $\varepsilon > 0$ such that $]x-\varepsilon, x+\varepsilon[\subset G$. Show that u is a topology on $\mathbb{R}$.

Or

Let (X, Y) be a topological space and A be a subset of X. Prove that the interior of A, A° is an open set.

(c) Prove that the space $\mathbb{C}^n$ is a Banach space.

Or

In an inner product space $(X, \langle \cdot, \cdot \rangle)$, if $x_n \to x$ and $y_n \to y$, then show that $\langle x_n, y_n \rangle \to \langle x, y \rangle$.

4. Answer the following questions : 10×3=30

(a) Prove that the metric space $(\mathbb{R}, d)$ is complete, where d is the usual metric on $\mathbb{R}$.

Or

Prove that all completions of a metric space are isometric.

(b) State and prove Baire's category theorem for metric spaces.

Or

Define uniformly continuous mapping in metric spaces. Give an example to show that a continuous mapping need not be uniformly continuous. Prove that the image of a Cauchy sequence under a uniformly continuous mapping is itself a Cauchy sequence. 1+3+6=10

(c) Prove that in a sequentially compact metric space, every open cover has a Lebesgue number.

Or

If f is a continuous mapping from a connected space X into $\mathbb{R}$, then prove that $f(X)$ is an interval.

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