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MATHEMATICS

(Major)

Paper : 6.4

(Discrete Mathematics)

Full Marks : 60

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer the following questions : $1 \times 7 = 7$

(a) State division algorithm of integers.

(b) Write the least positive integer of the form $172x + 20y$, $x, y \in \mathbb{Z}$.

(c) If p is a prime and $a \in \mathbb{Z}$, then show that $(a, p) = 1$ or $p | a$.

(d) State the converse of Fermat's theorem. Is it valid?

(e) Write the value of the sum $\sum_{d|n} \mu(d)$.

(f) State Chinese remainder theorem.

(g) What is the geometrical interpretation of a Diophantine equation $f(x, y) = 0$?

2. Answer the following questions : 2×4=8

(a) Show that there is no positive integer n such that $0 < n < 1$.

(b) Show that $\phi(n)$ is even if $n > 2$.

(c) State and prove the converse of Wilson's theorem.

(d) If x, y, z are primitive, positive, Pythagorean triple, then show that

$$\left(\frac{z-x}{2}, \frac{z+x}{2}\right) = 1$$

where x is odd.

3. Answer the following questions : 5×3=15

(a) Let $a, b \in \mathbb{Z}$, a or $b \neq 0$, $G = (a, b)$. Show that $G = ax_0 + by_0$ for some $x_0, y_0 \in \mathbb{Z}$.

Or

Show that every integer $n > 1$ can be expressed as a product of primes. Find the prime factorization of 40! 3+2=5

(b) Using Chinese remainder theorem, find the least positive integer which leaves the remainders 1, 6, 2 when divided by 7, 10, 11 respectively.

Or

Let $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ be the prime factorization of a positive integer $n > 1$. Show that the positive divisors of n are precisely the integers of the form $d = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$, where $0 \le a_i \le k_i$, $i = 1, 2, \cdots, r$.

(c) The linear congruence $ax \equiv b \pmod{m}$ has a solution if and only if $(a, m) | b$. Prove it.

Or

Show that the Diophantine equation $x^4 + y^4 = z^2$ has no solutions in positive integers.

4. Answer *either* (a) *or* (b) : 10

(a) (i) Let p be a prime and $\gcd(a, p) = 1$. Then show that the congruence $ax \equiv y \pmod{p}$ has a solution x_0, y_0 such that

$$0 < |x_0| < \sqrt{p}, \ 0 < |y_0| < \sqrt{p}$$ 5

(ii) If p_n is the nth prime number, then show that the sum

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n}$$

is never an integer. 5

(b) (i) Show that Euler's function ϕ is a multiplicative function. 7

(ii) Show that no prime of the form $4k+3$ is a sum of two squares. 3

5. Answer *either* (a) *or* (b) : 10

(a) (i) Give an example of an infinite Boolean algebra. In a Boolean algebra B, show that

$$a + a = a, \ a \cdot (a + b) = a, \ a, b \in B$$ 2+3=5

(ii) State and prove the 'principle of duality' in a Boolean algebra. Write down the dual of the proposition $a + b = 0 \Leftrightarrow a = 0, \ b = 0$. 4+1=5

(b) (i) Simplify the Boolean expression

$$(x + y)(x + z)(x' y')'$$

Draw a switching circuit which realizes the Boolean expression $x + y(z + x'(t + z'))$. 3+2=5 *(ii)* Construct the switching table for the switching function f represented by the Boolean expression $xyz + x'(y+z)$. 5

6. Answer *either* (a) or (b) : 10

(a) *(i)* Define 'logical equivalence'. Prove that if $\vDash A$ and $\vDash A \to B$, then $\vDash B$. 1+4=5

(ii) Construct the truth tables for NOR($\downarrow$) and NAND($\uparrow$). Show that $\{\wedge, \to\}$ is not an adequate system of connectives. 2+3=5

(b) *(i)* Using principle of substitution, show that if A, B be any two statement formulae, then

$$A \to B \equiv \sim B \to \sim A$$

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(ii) Assuming the truth value of $p \to q$ be T, construct the truth table for $(\sim p \wedge q) \leftrightarrow (p \vee q)$. 2

(iii) Define a truth function. Construct the truth function generated by the statement formula

$$\sim(\sim p \wedge q)$$

1+2=3

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