Total number of printed pages–16

3 (Sem–5/CBCS) MAT HC 1 (N/O)

2023

MATHEMATICS

(Honours Core)

OPTION–A

(For New Syllabus)

Paper : MAT–HC–5016

(Complex Analysis)

Full Marks : 60

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions : 1×7=7

   (a) Which point on the Riemann sphere represents ∞ of the extended complex plane $\mathbb{C} \cup \{\infty\}$?

   (b) A set $S \subseteq \mathbb{C}$ is closed if and only if S contains each of its ________ points. *(Fill in the gap)*

(c) Write down the polar form of the Cauchy-Riemann equations.

(d) The function $f(z) = \sinh z$ is a periodic function with a period ________ .

(Fill in the gap)

(e) Define a simple closed curve.

(f) Write down the value of the integral $\int_C f(z)\,dz$, where $f(z) = ze^{-2}$ and C is the circle $|z| = 1$.

(g) Find $\lim_{n\to\infty} z_n$, where $z_n = -1 + i\dfrac{(-1)^n}{n^2}$.

2. Answer the following questions : 2×4=8

(a) Let $f(z) = i\dfrac{z}{2}$, $|z| < 1$. Show that $\lim_{z\to 1} f(z) = \dfrac{i}{2}$, using $\varepsilon - \delta$ definition.

(b) Show that all the zeros of $\sinh z$ in the complex plane lie on the imaginary axis.

(c) Evaluate the contour integral $\int_C \dfrac{dz}{z}$, where C is the semi circle $z = e^{i\theta}$, $0 \le \theta \le \pi$

(d) Using Cauchy's integral formula, evaluate

$\int_C \dfrac{e^{2z}}{z^4}\,dz$, where C is the circle $|z| = 1$.

3. Answer ***any three*** questions from the following : 5×3=15

(a) Find all the fourth roots of −16 and show that they lie at the vertices of a square inscribed in a circle centered at the origin.

(b) Suppose $f(z) = u(x, y) + iv(x, y)$, $(z = x + iy)$ and $z_0 = x_0 + iy_0$, $w_0 = u_0 + iv_0$. Then prove the following :

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0,$$

$$\lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0; \text{ if and only}$$

if $\lim_{z\to z_0} f(z) = w_0$.

(c) (i) Show that the function $f(z) = Re\, z$ is nowhere differentiable.

(ii) Let $T(z) = \dfrac{az+b}{cz+d}$, where $ad - bc \neq 0$.

Show that $\lim\limits_{z \to \infty} T(z) = \infty$ if $c = 0$.

3+2=5

(d) Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant. Show that

$$\left| \int_C \frac{z+4}{z^3 - 1}\, dz \right| \leq \frac{6\pi}{7}$$

(e) State and prove fundamental theorem of algebra.

4. Answer ***any three*** questions from the following : 10×3=30

(a) (i) Show that $exp(z + \pi i) = -exp(z)$ 1

(ii) Show that

$$log(-1+i)^2 \neq 2\, log\,(-1+i)$$ 2

(iii) Show that

$$|\sin z|^2 = \sin^2 x + \sinh^2 y$$ 2

(iv) Show that a set $S \subseteq \mathbb{C}$ is unbounded if and only if every neighbourhood of the point at infinity contains at least one point of S. 5

(b) (i) Suppose that $f(z_0) = g(z_0) = 0$ and that $f'(z_0)$, $g'(z_0)$ exist with $g'(z_0) \neq 0$. Using the definition of derivative show that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$$ 5

(ii) Show that

$$z^2 e^{3z} = \sum_{n=2}^{\infty} \frac{3^{n-2}}{(n-2)!} z^n,$$

where $|z| < \infty$. 5

(c) State and prove Laurent's theorem.

(d) (i) Using definition of derivative, show that $f(z) = |z|^2$ is nowhere differentiable except at $z = 0$. 5

(ii) Define singular points of a function. Determine singular points of the functions :

$$f(z) = \frac{2z+1}{z(z^2+1)} \; ;$$

$$g(z) = \frac{z^3+i}{z^2-3z+2}$$ 1+4=5

(e) (i) Let $f(z) = u(x, y) + iv(x, y)$ be analytic in a domain D. Prove that the families of curves $u(x, y) = c_1$, $v(x, y) = c_2$ are orthogonal.

(ii) Let C denote a contour of length L and suppose that a function $f(z)$ is piecewise continuous on C. If M is a non-negative constant such that

$|f(z)| \leq M$ for all z in C

then show that

$$\left| \int_C f(z)\, dz \right| \leq ML.$$ 5+5=10

(f) (i) Prove that two non-zero complex numbers z_1 and z_2 have the same moduli if and only if $z_1 = c_1 c_2$, $z_2 = c_1 \bar{c}_2$, for some complex numbers c_1, c_2. 4

(ii) Show that mean value theorem of integral calculus of real analysis does not hold for complex valued functions $w(t)$. 3

(iii) State Cauchy-Goursat theorem. 1

(iv) Show that $\lim_{z \to \infty} \frac{z^2+1}{z-1} = \infty$. 2