Total number of printed pages–24

3 (Sem–6/CBCS) MAT HE 1/2/3/4

2024

MATHEMATICS

(Honours Elective)

Answer the Questions from any one Option.

OPTION – A

(Boolean Algebra and Automata Theory)

Paper : MAT–HE–6016

Full Marks : 80

Time : Three hours

OPTION – B

(Biomathematics)

Paper : MAT–HE–6026

Full Marks : 80

Time : Three hours

OPTION – C

(Mathematical Modeling)

Paper : MAT–HE–6036

Full Marks : 60

Time : Three hours

OPTION – D

(Hydromechanics)

Paper : MAT–HE–6046

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

Contd.

OPTION - A

(Boolean Algebra and Automata Theory)

Paper : MAT–HE–6016

1. Answer the following questions : 1×10=10

   (a) When is an ordered set said to be chain ?

   (b) Define dual of an ordered set.

   (c) Give an example of a lattice without a zero element.

   (d) Define lattice isomorphism.

   (e) Define a modular lattice.

   (f) When are two Boolean polynomials said to be equivalent ?

   (g) What is an alphabet in automata theory ?

   (h) If $\Sigma = \{0, 1\}$, find Σ^0.

   (i) Define regular language in automata theory.

   (j) Write two palindromes from $\Sigma = \{0, 1\}$.

2. Answer the following questions : 2×5=10

   (a) Prove that a singleton subset of a lattice L is a sublattice of L.

   (b) Give an example of a poset which has exactly one maximal element but does not have a greatest element. Justify your answer.

   (c) Give an example of a lattice which is not a Boolean algebra. Justify your answer.

   (d) Define DFA.

   (e) Write two trivial languages for any alphabet Σ.

3. Answer ***any four*** questions from the following : 5×4=20

   (a) Define bottom and top elements of an ordered set. Give an example of an ordered set which have both bottom and top elements. Write both the elements. Give another example of an ordered set which does not have both bottom and top elements. 2+1+1+1=5

   (b) Show that the set N (natural numbers), ordered by divisibility is a distributive lattice. Is it complemented ?

(c) For any two elements x, y in a lattice L, prove that the interval

$$[x, y] = \{a \in L \mid x \leq a \leq y\}$$

is a sublattice of L.

(d) Let B be a finite Boolean algebra, and let A denote the set of all atoms in B. Then prove that

$$(B, \wedge, V) \cong_b (P(A), \cap, \cup)$$

(e) Draw the diagram of the switching circuit $p = x_1(x_2(x_3 + x_4) + x_3(x_5 + x_6))$.

(f) Prove that if L and M are regular languages, then so are $L \cup M$ and $L - M$.

4. Answer the following questions : 10×4=40

(a) (i) Let P be a set on which a binary relation $<$ is defined such that, for all $x, y, z, \in P$,

A. $x < x$ is false

B $x < y$ and $y < z$ imply $x < z$

Prove that if $\leq$ is defined by

$$x \leq y \Leftrightarrow (x < y \text{ or } x = y)$$

then $\leq$ is an order on P. 5

(ii) Let P and Q be ordered sets. Prove that $(a_1, b_1) \prec (a_2, b_2)$ in $P \times Q$ if and only if $(a_1 = a_2, b_1 \prec b_2)$ or $(a_1 \prec a_2, b_1 = b_2)$. ($\prec$ is a covering relation). 5

Or

(i) Let P and Q be ordered sets. Prove that P and Q are order-isomorphic if and only if there exist order-preserving maps $\phi : P \to Q$ and $\psi : Q \to P$ such that $\phi o \psi = id_Q$ and $\psi o \phi = id_P$, where $id_S : S \to S$ such that $id_S(x) = x, \forall x \in S$. 5

(ii) Let $X = (1, 2, \ldots n\}$ and define

$\phi : P(X) \to 2^n$ by

$\phi(A) = (\varepsilon_1, \varepsilon_2, \ldots \varepsilon_n)$, where

$$\varepsilon_i = 1, i \in A$$

$$= 0, i \notin A$$

Prove that ϕ is an order-isomorphism. 5

(b) (i) Let $(L, \leq)$ be an ordered set. Let $x \vee y = sup(x,y)$ and $x \wedge y = inf(x,y)$. Then prove that $(L, \wedge, \vee)$ is a lattice. 5

(ii) Let L be a lattice, if $a, b, c \in L$, then prove that

$a \leq b \Rightarrow a \vee c \leq b \vee c$ and $a \wedge c \leq b \wedge c$. 5

Or

(i) Prove that a lattice L is distributive if and only if

$(x \vee y = x \vee z, x \wedge y = x \wedge z) \Rightarrow y = z$. 5

(ii) Let L, K be two lattices. Let $f : L \to K$ be an isomorphism. Then prove that f^{-1} is also an isomorphism. 5

(c) (i) Prove that in a Boolean algebra $B, x \leq y \Leftrightarrow x' \geq y'$. 5

(ii) Let L be a lattice. Prove that if L is a Boolean algebra, then L is relatively complemented. 5

Or

(i) Find the disjunctive normal form of

$$((x_1 + x_2)' x_1 + x_2''')' + x_1 x_2 + x_1 x_2'$$

5

(ii) Write the minimal form of p whose disjunctive normal form is given by

$$wxyz' + wxy'z' + wx'yz + wx'yz'$$
$$+ w'x'yz + w'x'yz' + w'x'y'z$$

5

(d) (i) Design a DFA to accept the language $L = \{w \mid w$ has both an even number of 0's and an even number of 1's$\}$. 5

(ii) If $D = (Q_D, \Sigma, \delta_D, \{q_0\}, F_D)$ is the DFA constructed from NFA, $N = (Q_N, \Sigma, \delta_N, q_0, F_N)$ by the subset construction, then prove that $L(D) = L(N)$. 5

Or

(i) Prove that a language L is accepted by some ε-NFA only if L is accepted by some DFA. 5

(ii) If $L = L(A)$ for some DFA, A, then there is a regular expression R such that $L = L(R)$. 5

OPTION – B

Paper : MAT–HE–6026

(Biomathematics)

1. Answer the following questions : 1×10=10

   (a) Give an example of nonhomogeneous linear differential equation.

   (b) The difference equation

   $x_{t+1} = tx_t + 2t^2 x_{t-1} + sin(t)$ is a

   (i) second order linear difference equation

   (ii) non-autonomous equation

   (iii) non-homogeneous equation

   (iv) All of the above

   (Choose the correct answer)

   (c) Write the condition that an equilibrium $\bar{x}$ of $x_{t+1} = f(x_t)$ is said to be hyperbolic.

   (d) Logistic growth is often referred to as ______ growth. *(Fill in the blank)*

   (e) Let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

   What are the eigenvalues of the matrix A ?

   (f) Write the characteristic equation for the matrix

   $$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

   (g) Write an equilibrium solution of the system

   $$\frac{dx}{dt} = a_{11}x + a_{12}y$$

   $$\frac{dy}{dt} = a_{21}x + a_{22}y$$

   (h) Write one use of Routh-Hurwitz criteria.

   (i) Arrange the following relation in correct order

(1) Anderson and May	*(i)* Age-structure model
(2) Kermack and McKendrik	*(ii)* Predator-Prey model
(3) Patrick Holt Leslie	*(iii)* Epidemic model
(4) Alfred Lotka and Vito Volterra	*(iv)* Measles model

   (j) What is a saddle point ?

2. Answer the following questions : 2×5=10

(a) Write two basic requirements of a mathematical problem to be properly posed.

(b) What is a Leslie matrix ?

(c) Define a spectral radius of a $k \times k$ matrix A.

(d) Find the general solution to the equation

$$x_{t+2} - 16x_t = 0$$

(e) Use an integrating factor to find the unique solution to the initial value problem

$$\frac{dx}{dt} - 3t^2 x = 4te^{-t^3},\ x(0) = 1$$

3. Answer ***any four*** questions : 5×4=20

(a) Let the Leslie matrix $L = \begin{pmatrix} b_1 & b_2 \\ s_1 & 0 \end{pmatrix}$ where $b_2 > 0$ and $s_1 > 0$. Show that L is primitive.

(b) Find the eigenvalues and eigenvectors of matrix A given by

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$$

then find the general solution to

$$X(t+1) = A \times (t)$$

(c) Find all the equilibria for the system of difference equation

$$x_{t+1} = \frac{ax_t y_t}{1 + x_t},\ a > 0$$

$$y_{t+1} = \frac{bx_t y_t}{1 + y_t},\ b > 0$$

(d) Show that the solution approaches to zero of the linear differential equation

$$\frac{d^3x}{dt^3} + 4\frac{d^2x}{dt^2} - \frac{dx}{dt} + ax = 0$$

(e) With a suitable example explain briefly phase plane analysis.

(f) Show that the zero equilibrium is always unstable if

$$\frac{dx}{dt} = ax + y,\ a \neq -1$$

$$\frac{dy}{dt} = -x + y$$

When is the zero equilibrium a saddle point ?

4. Answer the following questios : 10×4=40

(a) Write a brief note on the development of logistic growth model.

Or

State briefly a measles model with vaccination.

(b) Show that the 2-cycle

$$\bar{x}_{1,2} = \frac{1 \pm \sqrt{4r-3}}{2}$$

of the difference equation

$x_{t.1} = r - x^2$ is locally asymptotically stable if $\frac{3}{4} < r < \frac{5}{4}$ and unstable if $r > \frac{5}{4}$.

Or

Show that the difference equation

$$x_{t+1} - a x_t^3 = f(x_t), a > 0,$$

$f : [0, \infty) \to [0, \infty)$ has no 2-cycle on the interval $[0, \infty)$.

(c) Suppose the coefficients of the characteristic polynomial are real. If all the roots of the characteristic polynomial

$$P(\lambda) = \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \ldots + a_n$$

are negative or have negative real parts, then prove that the coefficients $a_i > 0$ for $i = 1, 2 \ldots n$.

Or

Show that the solution to the pharmacokinetics model is

$$x(t) = \frac{1}{a}\left(1 - e^{-at}\right)$$

$$y(t) = \frac{1}{b} + \frac{e^{-at}}{a-b} - \frac{ae^{-bt}}{b(a-b)}$$

(d) For the following differential equation, find the equilibrium.

$$\frac{dx}{dt} = x(a-x)(x-b)^2, \; 0 < a < b$$

Then graph the phaseline diagram. Use the phaseline diagram to determine the stability of equilibrium.

Or

Discuss briefly SIR epidemic model with a suitable example.

OPTION - C

(Mathematical Modelling)

Paper : MAT-HE-6036

1. Answer the following questions : 1×7=7

(a) When does an improper integral over an infinite interval $\int_a^\infty g(t)dt$ converge ?

(b) Write Bessel's equation of order n.

(c) State the linearity property of Laplace transform.

(d) What is meant by two term recurrence relation related to power series solution.

(e) Find Laplace transform of $f(t) = t^n$.

(f) Name two high-level simulation languages used in constructing a simulation model.

(g) Write down the power series representation of the Taylor series with centre $x = a$.

2. Answer the following questions : 2×4=8

(a) Find the inverse Laplace transform of

$$F(s) = \frac{3}{s(s+5)}$$

(b) The recurrence relation for the power series solution $\sum_{n=0}^{\infty} C_n x^n$ of a differential equation is obtained as

$$C_{n+1} = \frac{n+2}{3(n+1)} C_n \text{ for } x \geq 0.$$

Find the radius of convergence ρ.

Also write the condition of convergence of the power series.

(c) Find the exponents in the possible Frobenius series solution of

$$2x^2y'' + 3xy' - (x^2+1)y = 0.$$

(d) Find the values of $\Gamma 5$ and $\Gamma \frac{5}{2}$.

3. Answer ***any three*** questions of the following : 5×3=15

(a) Solve the initial value problem

$$x'' - x' - 6x = 0, \; x(0) = 2, \; x'(0) = -1$$

using Laplace and inverse Laplace technique.

(b) Define singular points of a linear differential equation. Mention different types of singular points. Investigate the nature of the point $x=0$ for the differential equation

$$x^4y'' + x^2 \sin x\, y' + (1 - \cos x)y = 0.$$

1+1+3=5

(c) What is meant by random numbers in Monte Carlo simulation? Use linear congruence method to generate 8 random numbers using

$a = 5$, $b = 1$ and $c = 8$ and taking initial seed $x_0 = 17$. 1+4=5

(d) Find a power series solution of the differential equation $y' = 4y$.

(e) Find a Frobenius solution of Bessel's equation of order zero

$$x^2y'' + xy' + x^2y = 0$$

Also deduce the Bessel function of order zero of first kind.

4. Answer **either** (a) **or** (b) and (c) :

(a) Solve the system

$2x'' = -6x + 2y$

$y'' = 2x - 2y + 40 \sin 3t$

subject to the initial conditions

$x(0) = x'(0) = y(0) = y'(0) = 0$ 10

Or

(b) Solve the initial value problem

$$x'' + 9x = 1, \quad x(0) = 0 = x'(0)$$ 6

(c) Evaluate $L^{-1}\left\{\dfrac{s+2}{s^2 - 2s + 5}\right\}$ 4

5. Answer **either** (a) **or** (b) :

(a) Find *two* linearly independent Frobenius series solutions of the equation

$$4xy'' + 2y' + y = 0 \quad (x > 0)$$ 10

(b) Find the general solution in powers of x of the equation

$$(x^2 - 4)y'' + 3xy' + y = 0$$

Then find the particular solution with

$$y(0) = 4, \; y'(0) = 1$$ 10

6. Answer **either** (a) **or** (b) :

(a) A small harbor has unloading facilities for ships. Only one ship can be unloaded at any time. The unloading time required for a ship depends on the type and amount of cargo and varies from 45 to 90 minutes. 10

Below is given a situation with 5 ships :

	Ship 1	Ship 2	Ship 3	Ship 4	Ship 5
Time between successive ships (in minutes) :	20	30	15	120	25
Unloading time :	55	45	60	75	80

(i) Draw the timeline diagram depicting clearly the situation for each ship, the idle time for harbor and the waiting time.

(ii) Identify the ship having highest waiting time.

(iii) Identify the ship which have to stay highest time in the harbor.

(iv) Find the average waiting time for the ships. 7+1+1+1=10

Or

(b) Using Monte Carlo simulation, describe the process and also write an algorithm to calculate the area under the curve of the continuous function $y = f(x)$ satisfying $0 \le f(x) \le M$ over the closed interval $a \le x \le b$. 10

OPTION - D

Paper : MAT-HE-6046

(Hydromechanics)

1. Answer the following questions : 1×10=10

(a) State Boyle's law.

(b) Define absolute zero temperature.

(c) What do you mean by resultant vertical thrust on any surface of homogeneous liquid at rest under the action of gravity ?

(d) What is an adiabatic change ?

(e) Define the term 'convective equilibrium' for a gas.

(f) Define thermal conductivity.

(g) Distinguish between a Newtonian fluid and a non-Newtonian fluid.

(h) Velocity potential exists only for an irrotational motion. *(Write True **or** False)*

(i) For an irrotational flow

(i) div $\vec{q} = 0$

(ii) div $\vec{q} \neq 0$

(iii) curl $\vec{q} = 0$

(iv) curl $\vec{q} \neq 0$

(Choose the correct option)

(j) The equation of streamline in vector form is

(i) $\vec{q} \times d\vec{r} = 0$

(ii) $\vec{q}.d\vec{r} = 0$

(iii) $\vec{r}.d\vec{q} = 0$

(iv) None of the above

(Choose the correct answer)

2. Answer the following questions : 2×5=10

(a) Obtain the differential equation of lines of force at any point (x, y, z).

(b) Define surface of buoyancy and surface of floatation.

(c) Prove that the position of the center of pressure (CP) of a plane area is independent of the inclination of the area to the vertical.

(d) Derive Bernoulli's theorem for a steady motion.

(e) If the velocity $\vec{q}$ of a fluid particle is given by $\vec{q} = x\hat{i} - y\hat{j}$, determine the equations of streamlines.

3. Answer the following questions : ***(any four)*** 5×4=20

(a) The component of forces X, Y, Z acting on an element of fluid at (x, y, z) are given by

$$X = y^2 + 2\lambda yz + z^2$$

$$Y = z^2 + 2\mu zx + x^2$$

$$Z = x^2 + 2\nu xy + y^2$$

Show that if equilibrium be possible, then $2\lambda = 2\mu = 2\nu = 1$.

(b) If $X = y(y+z)$, $Y = z(z+x)$ and $Z = y(y-x)$. Prove that surfaces of equal pressure are hyperbolic paraboloid $y(x+z) = c(y+z)$.

Then prove that the curves of equal pressure and density are given by

$$y(x+z) = \text{constant}$$

$$y + z = \text{constant}$$

(c) A mass M of a gas of uniform temperature is diffused through all spaces, and at point (x, y, z), the components of forces per unit mass are

(j) The equation of streamline in vector form is

(i) $\vec{q} \times d\vec{r} = 0$

(ii) $\vec{q}.d\vec{r} = 0$

(iii) $\vec{r}.d\vec{q} = 0$

(iv) None of the above

(Choose the correct answer)

2. Answer the following questions : 2×5=10

(a) Obtain the differential equation of lines of force at any point (x, y, z).

(b) Define surface of buoyancy and surface of floatation.

(c) Prove that the position of the center of pressure (CP) of a plane area is independent of the inclination of the area to the vertical.

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3. Answer the following questions : ***(any four)*** 5×4=20

(a) The component of forces X, Y, Z acting on an element of fluid at (x, y, z) are given by

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Show that if equilibrium be possible, then $2\lambda = 2\mu = 2\nu = 1$.

(b) If $X = y(y+z)$, $Y = z(z+x)$ and $Z = y(y-x)$. Prove that surfaces of equal pressure are hyperbolic paraboloid $y(x+z) = c(y+z)$.

Then prove that the curves of equal pressure and density are given by

$$y(x+z) = \text{constant}$$

$$y+z = \text{constant}$$

(c) A mass M of a gas of uniform temperature is diffused through all spaces, and at point (x, y, z), the components of forces per unit mass are

$-Ax, -By, -Cz$. The pressure and density at the origin are P_0 and ρ_0 respectively. Prove that $ABC\rho_0 M^2 = 8\pi^3 P_0{}^3$.

(d) A mass of fluid moves in such a way that each particle describes a circle in one plane about a fixed axis. Show that the equation of continuity is $\frac{\partial p}{\partial t} + \frac{\partial(\rho\omega)}{\partial\theta} = 0$.

(e) For an incompressible fluid, $\bar{q} = [-wy,\ wx,\ 0]$, w = constant.

Discuss the nature of the flow.

(f) Show that the acceleration $\vec{a}$ of a fluid particle of fixed density can be expressed as a material derivative of the velocity vector $\vec{q}$.

4. Answer the following questions : 10×4=40

(a) A vessel full of water is in the form of an eight part of an ellipsoid (axis a, b, c) boundary by the three principal axes. The axis c is vertical and the atmospheric pressure is neglected. Prove that the resultant fluid pressure on the curved surface is a force of intensity

$$\frac{1}{3}pg\left[b^2c^4 + a^2c^4 + \frac{1}{4}\pi^2a^2b^2c^2\right]^{\frac{1}{2}}$$

Or

(b) Prove that the thrust on a heavy homogeneous liquid on a plane area is equal to the product of the area and the pressure at the center of gravity.

(c) A right circular cylinder, open at the top, is filled with water and the whole of it revolves with angular velocity ω about the axis. If not more than half of the water is split, prove that the thrust on the base is

$$\Pi\rho g a^2 h\left(1 - \frac{a^2\omega^2}{4gh}\right)$$

where h is the height, a is the radius of the base of the cylinder and ρ is the density of the water.

Or

(d) A closed cylinder, very nearly filled with liquids, rotates uniformly about a generating line, which is vertical. Find the resultant pressure on the curved surface. Also, determine the point of action of the pressure on its upper end.

(e) Determine the equation of continuity for an incompressible fluid in vector method. Interpret the result physically.

Or

(f) Determine the constants l, m, n such that the velocity $\vec{q}$, given by

$$\vec{q} = \{(x + lr)\hat{i} + (y + mr)\hat{j} + (z + mr)\hat{k}\} / r(x + r)$$

where $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$ may satisfy the equation of continuity for a liquid.

(g) Describe Lagrange's and Eulerian methods for describing the fluid flows and distinguish between them.

Or

(h) The velocities of an incompressible fluid at the point (x, y, z) is given by

$$\left(\frac{3xz}{r^5}, \frac{3yz}{r^5}, \frac{3z^2 - r^2}{r^5},\right)$$

Prove that the liquid motion is possible. Find the velocity potential and streamlines.
